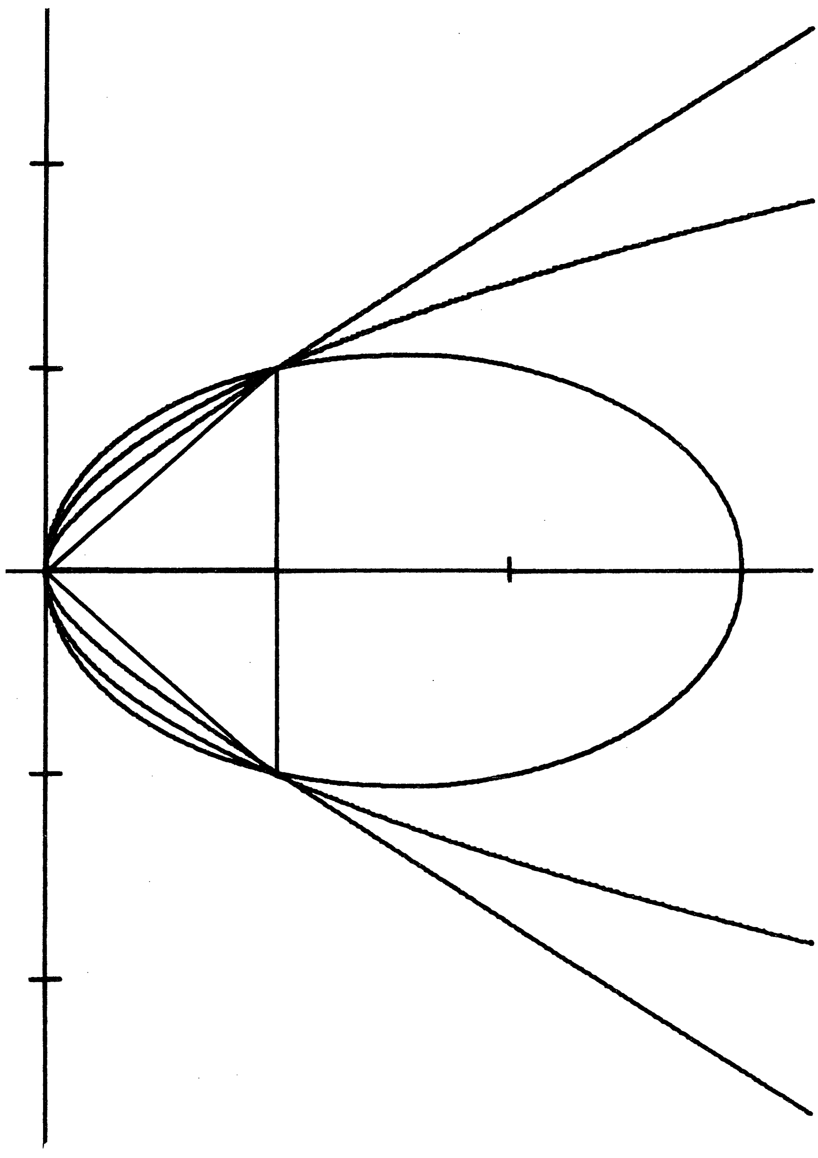
MATHEMATICS MAGAZINE



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September 1984

COMPUTING CATALAN NUMBERS • STIRLING IDEAS
ARCHIMEDES REVISITED • DIFFERENTIATION TEST

MATHEMATICS MAGAZINE

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COVER: Archimedes revisited. See p. 224 and p. 242. Design by Gary Eichelsdorfer.

EDITORIAL POLICY

Mathematics Magazine is a journal which aims to provide inviting, informal mathematical exposition. Manuscripts for the *Magazine* should be written in a clear and lively expository style and stocked with appropriate examples and graphics. Our advice to authors is: say something new in an appealing way or say something old in a refreshing way. The *Magazine* is not a research journal and so the style, quality, and level of articles should realistically permit their use to supplement undergraduate courses. The editor invites manuscripts that provide insight into the history and application of mathematics, that point out interrelationships between several branches of mathematics and that illustrate the fun of doing mathematics.

The full statement of editorial policy appears in this *Magazine,* Vol. 54, pp. 44-45, and is available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

Send new manuscripts to: Doris Schattschneider, Editor, Mathematics Magazine, Moravian College, Bethlehem, PA 18018. Manuscripts should be typewritten and double spaced and prepared in a style consistent with the format of *Mathematics Magazine*. Authors should submit the original and one copy and keep one copy. Illustrations should be carefully prepared on separate sheets in black ink, the original without lettering and two copies with lettering added.

AUTHORS

Douglas M. Campbell ("The Computation of Catalan Numbers") is a native of California who did his undergraduate work at Harvard, took two years off for a mission to Hawaii for The Church of Jesus Christ of Latter-day Saints (Mormon), obtained his Ph.D. at the University of North Carolina Chapel Hill, and finally settled in Utah.

He has taught at Brigham Young University since 1971, where he recently changed from the mathematics to the computer science department.

An avid fisherman, he has also been a visiting professor at the University of Maryland and the University of Michigan. His early research was in univalent and normal analytic functions, and his present interests are in computibility theory, human factors, and system design.

ILLUSTRATIONS

Vic Norton portrays Jill and Peter, as well as some memorable moments in history for the article *The Computation of Catalan Numbers.*

Gary Eichelsdorfer provided the computer-drawn graphs on p. 212 and 242.

All other illustrations were provided by the authors.

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The Computation of Catalan Numbers

An opportunity for a mathematics and a computer science student to have a dialogue.

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Students in mathematics and computer science often confront an identical problem in different contexts. In many cases, each can profit by examining such a problem from both viewpoints. This paper illustrates this theme by means of the **Catalan numbers**, defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}, \qquad n = 0, 1, \ldots. \tag{1}$$

The first six of these are: $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$.

A mathematics student often deals with numbers without reference to a concrete problem. Mathematicians, therefore, often distinguish between a number such as π^2 and a closed expression such as $6(1 + 2^{-2} + 4^{-2} + \cdots)$ which defines that number. To a mathematician, a number is an abstract entity, independent of representation.

A computer science student often evaluates closed expressions arising from concrete problems. Computer scientists, therefore, often focus on the concrete problem and may not see the numbers which arise as having their own inherent structure and nature.

Catalan numbers are everywhere

The Catalan numbers are ubiquitous. Here are several of the areas in which they arise.

The Catalan number C_n is the number of ways the $n+1$ terms of the sequence $x_1, \ldots, x_{n+1}$ may be combined (without interchanging the order of the subscripts) by a binary nonassociative product [**1**], [**11**]. For example, for $n = 3$:

$$x_1((x_2x_3)x_4), \quad (x_1x_2)(x_3x_4), \quad ((x_1x_2)x_3)x_4,$$
$$x_1(x_2(x_3x_4)), \quad (x_1(x_2x_3))x_4.$$

The Catalan number C_n is the number of ways of moving from the point $(0,0)$ to the point $(2n+2, 0)$ in the xy-plane, never touching the x-axis (except at start and finish), with diagonal steps which join (x, y) to $(x+1, y+1)$ or (x, y) to $(x+1, y-1)$ [**4**, Chapter 3]. For $n = 3$, the five paths possible from $(0,0)$ to $(8,0)$ are shown in FIGURE 1.

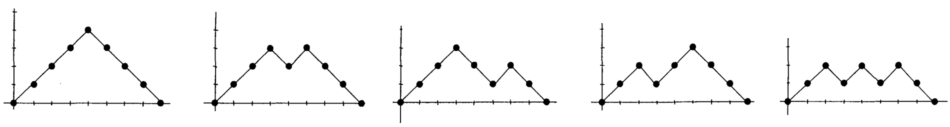


FIGURE 1

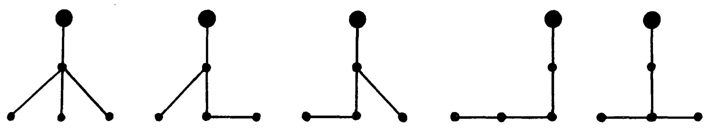


FIGURE 2

The Catalan number C_n is the number of planar connected graphs without loops having $n+2$ vertices and $n+1$ edges and a distinguished vertex of valence one [3], [7]. The five such graphs for $n=3$ are shown in FIGURE 2.

The Catalan number C_n is the number of ways of decomposing a (labeled) convex $(n+2)$-gon into triangles using $n-1$ nonintersecting diagonals (Euler, 1786), [9]. FIGURE 3 shows the five possible such dissections of a pentagon into triangles.

The Catalan number C_n is the number of $(n-1)$-tuples $(a_1, a_2, \ldots, a_{n-1})$ of positive integers such that in the sequence

$$1, a_1, a_2, \cdots, a_{n-1}, 1,$$

every a_i divides the sum of its two neighbors [12, p. 22]. For example, for $n=3$, there are five such pairs of positive integers:

$$(1,1) \quad (1,2) \quad (2,1) \quad (2,3) \quad (3,2).$$

The Catalan number C_n is the number of sequences containing $n+1$ terms equal to 1 and $n+1$ terms equal to -1 such that all partial sums (except the last) are positive [3]. For $n=3$, here are the five sequences of length eight that satisfy the stated conditions:

$$1,1,-1,1,-1,1,-1,-1; \quad 1,1,1,1,-1,-1,-1,-1; \quad 1,1,1,-1,-1,1,-1,-1;$$
$$1,1,1,-1,1,-1,-1,-1; \quad 1,1,-1,1,1,-1,-1,-1.$$

There are three other areas in which Catalan numbers arise that are of special interest to the computer science student. The set of **well-formed sequences of parentheses** is defined recursively as follows: the empty string is a well-formed sequence of parentheses; if A is a well-formed sequence of parentheses, then so is (A); if A and B are well-formed sequences of parentheses, then so is AB. No sequence of parentheses is well-formed unless it can be derived by a finite number of applications of these rules. The Catalan number C_n is the number of well-formed strings of parentheses of length 2n [5]. The five well-formed strings of parentheses of length 6 (i.e., $n=3$) are as follows:

()()(), ((())), (())(), ()(()), (()()).

A **stack** is a data structure used in a computer. A stack in a computer acts like a stack of trays in a cafeteria; only the top tray is accessible.

Consider the ordered sequence of elements $e_1, e_2, \ldots, e_n$ in stack A with e_1 on the top and e_n on the bottom, while stacks B and C are initially empty. Suppose the only legal moves are: the top element of A can be moved to stack B and the top element of stack B can be moved to stack C. After exactly $2n$ such moves, stacks A and B are empty and stack C contains a permutation of the elements $e_1, \ldots, e_n$. The Catalan number C_n is the number of permutations of the elements

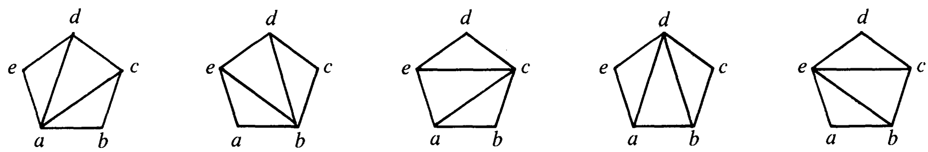


FIGURE 3

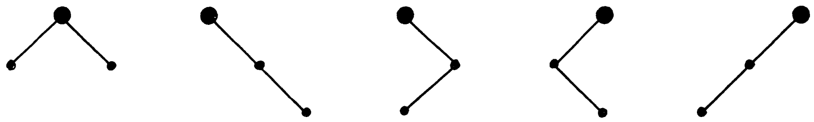


FIGURE 4

$e_1, \ldots, e_n$ that can be created in stack C according to these rules [6]. For $n = 3$, the five permutations of (1,2,3) that can be achieved are:

$$(1,2,3) \quad (3,2,1) \quad (2,3,1) \quad (1,3,2) \quad (3,1,2).$$

A sequence of moves that creates the last permutation is shown below.

e_1																				e_3
e_2			e_2				e_2							e_1			e_1			e_1
e_3			e_3	e_1		e_3	e_1		e_3	e_1	e_2	e_3		e_2		e_3	e_2			e_2
A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C

(The reader might wish to verify that the permutation (2,1,3) can *not* be so generated.)

A **binary tree** is defined recursively as a finite set of nodes which is either empty, or consists of a distinguished node called a root and two disjoint binary trees called the left and right subtrees of the root. The Catalan number C_n is the number of binary trees of n nodes [8]. The five binary trees having 3 nodes are shown in FIGURE 4.

Calculating small Catalan numbers

We now turn to the problem of calculating Catalan numbers. Imagine a mathematics student named Peter with minimal computing experience but unlimited faith in the power of the computer, and a computer science student named Jill with minimal mathematical background but unlimited faith in mathematical symbolism, who set out to calculate as many of the Catalan numbers as possible on an Apple II computer with UCSD Pascal.

P. Well, Jill, let's see what you can do with the "long integers" in UCSD Pascal. Since the Apple will do exact arithmetic on integers with less than 37 digits, I'd like to see some big Catalan numbers.

J. I assume that you want me to compute them directly from your representation:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)n!n!}. \tag{1}$$

I'll compute $(2n)!$ and $n!$, square $n!$, multiply by $(n+1)$ and divide the result into $(2n)!$ Here we go.

P. Well, that was disappointing. It stopped with C_{16} as 35,357,670 but it won't compute C_{17}. Why? I thought you said that the long integers would let us do 36-digit Catalan numbers, but C_{16} is a measly eight digits!

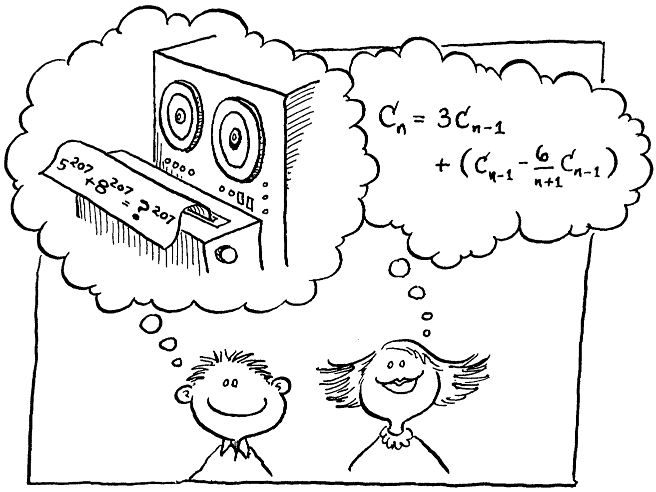
J. Gimme a minute. Ah here's the problem. For $n = 17$, the quantity $(2n)!$ has more than 36 digits. Looks like the intermediate calculations grow too fast. A classical computer science problem: an algorithm of limited use because of intermediate calculation blowup.

P. Let's use some elementary properties of the factorial to reduce the excessively large numbers generated by $(2n)!$. Since $(2n)! = (2n)(2n-1)\cdots(n+1)n!$, we can rewrite C_n as:

$$C_n = \frac{(2n)(2n-1)\cdots(n+2)}{n!}, \qquad n = 2,3,\ldots. \tag{2}$$

J. Fine. I'll multiply $(n+2)(n+3)\cdots(2n-1)(2n)$ and then divide by $n!$. Here goes.

P. That's better. Now we get up to $C_{24} = 1{,}289{,}904{,}147{,}324$ correctly, but that has only thirteen digits. What's wrong now? I thought you said these things were powerful.



J. They are powerful! Maybe it's the way you're telling me to calculate the numbers. The numerator $(2n)(2n-1)\cdots(n+2)$ has more than 36 digits when n is 25. Are you sure that you have a decent representation for the Catalan numbers?

P. Representation? What do you mean? A number *is*. A number is independent of its representation.

J. A number may be independent of its representation, but there are different ways of computing the value of a number and you have already seen that representation (2) has advantages over representation (1). Maybe we can compute C_n in terms of C_{n-1}. Let's see. Since

$$\begin{aligned} C_n &= \frac{(2n)!}{(n+1)n!n!} = \frac{(2n)(2n-1)(2n-2)!}{(n+1)(n)(n)(n-1)!(n-1)!} \\ &= \frac{(2n)(2n-1)}{(n+1)(n)}C_{n-1} = \frac{2(2n-1)}{(n+1)}C_{n-1}, \end{aligned} \tag{3}$$

we can compute C_n by multiplying the previous value of C_{n-1} by $2(2n-1)$ and then dividing by $n+1$.

P. That is so simple that even I can write a program for that. Here goes.

J. At last, a big Catalan number,

$$C_{61} = 6{,}182{,}127{,}958{,}584{,}855{,}650{,}487{,}080{,}847{,}216{,}336,$$

a number with 34 digits. Let me check something. Yes, the quantity $2(2n-1)\ C_{n-1}$ has more than 36 digits for $n = 62$.

P. Well, that's it then. Thanks.

J. Wait. Don't you want more?

P. Look, I'm not greedy; you got a 34 digit number and there is a 36 digit limitation. That's nearly optimal. There's not much more for me to learn by tinkering with a few more digits.

J. Maybe, but I notice that the reason for the overflow is that you're multiplying C_{n-1} by $2(2n-1)$ which grows like $4n$ while C_n is actually C_{n-1} times $2(2n-1)/(n+1)$ which acts like 4, not like $4n$. Let's rewrite C_n as:

$$\begin{aligned} C_n &= \frac{2(2n-1)}{n+1}C_{n-1} \\ &= 3C_{n-1} + \left(C_{n-1} - \frac{6}{n+1}C_{n-1}\right), \qquad n = 1,2,\ldots \end{aligned} \tag{4}$$

and compute C_n by multiplying C_{n-1} by 6, dividing by $n+1$, subtracting this number, which is *smaller* than C_{n-1}, from C_{n-1} and then adding the result to 3 times C_{n-1}.

P. Well,, after you play with that I'll show you why you need to review your elementary algebra! But go ahead. Hmmm. You got

$$C_{64} = 368{,}479{,}169{,}875{,}816{,}659{,}479{,}009{,}042{,}713{,}546{,}950$$

—a number with 36 digits. Impressive. But aren't you aware that $3C_{n-1} + C_{n-1}$ equals $4C_{n-1}$?

J. Of course! But you missed the point of reducing overflow. If we were to write C_n as $4C_{n-1} - (6C_{n-1}/(n+1))$ then we would calculate 4 times C_{n-1} and *subtract* something. Thus, the computation may terminate not because the Catalan number C_n is too big, but because the intermediate calculation $4C_{n-1}$ is too big. If we write C_n as $3C_{n-1} + [C_{n-1} - (6C_{n-1}/(n+1))]$, then we will add two positive numbers each of which is smaller than C_n. That way the only time we get an overflow is when C_n, and not some intermediate calculation, is too big.

P. Tricky! I should have thought of that. But how did you know that $(n+1)$ divides $6C_{n-1}$ without a remainder? It's not obvious that

$$\frac{6}{n+1}C_{n-1} = \frac{6}{n(n+1)}\binom{2n-2}{n-1}$$

is always an integer.

J. Actually, I wasn't sure and I had put in a line to print an error message if $6C_{n-1}/(n+1)$ left a remainder. But no error was printed for n from 1 to 64.

P. Sixty-four cases is hardly a proof! On the other hand, it is evidence. Let me try a proof by induction.

(Half an hour later, Peter returns.)

P. Well, induction didn't work, but at least I found a proof that does work. We can always write $m!$ as

$$\prod_{p \text{ prime}} p^{e_p}, \text{ where } e_p = \sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]$$

and where $[x]$ is the integral part of x. Thus

$$\frac{C_n}{n+2} = \frac{(2n)!}{(n+2)!n!} = \prod_p p^{f_p}, \text{ where } f_p = \sum_{k=1}^{\infty}\left[\frac{2n}{p^k}\right] - \left[\frac{n+2}{p^k}\right] - \left[\frac{n}{p^k}\right].$$

If we write

$$\frac{n}{p^k} = \left[\frac{n}{p^k}\right] + \delta, \qquad 0 \leqslant \delta < 1,$$

we see that

$$\frac{2n}{p^k} = 2\left[\frac{n}{p^k}\right] + 2\delta, \qquad \frac{n+2}{p^k} = \left[\frac{n}{p^k}\right] + \delta + \frac{2}{p^k},$$

and therefore

$$\left[\frac{2n}{p^k}\right] = 2\left[\frac{n}{p^k}\right] + [2\delta], \qquad \left[\frac{n+2}{p^k}\right] = \left[\frac{n}{p^k}\right] + \left[\delta + \frac{2}{p^k}\right].$$

J. Then, the only time that $n+2$ does not divide C_n is when some exponent f_p is negative.

P. Correct. Therefore, let's examine the terms which make up f_p:

$$\left[\frac{2n}{p^k}\right] - \left[\frac{n+2}{p^k}\right] - \left[\frac{n}{p^k}\right] = 2\left[\frac{n}{p^k}\right] + [2\delta] - \left(\left[\frac{n}{p^k}\right] + \left[\delta + \frac{2}{p^k}\right]\right) - \left(\left[\frac{n}{p^k}\right]\right)$$

$$= [2\delta] - \left[\delta + \frac{2}{p^k}\right].$$

In order for some f_p to be negative, at least one of its individual terms,

$$\left[\frac{2n}{p^k}\right] - \left[\frac{n+2}{p^k}\right] - \left[\frac{n}{p^k}\right]$$

must be negative. But when $1/2 \leqslant \delta \leqslant 1$,

$$\left[\frac{2n}{p^k}\right] - \left[\frac{n+2}{p^k}\right] - \left[\frac{n}{p^k}\right] = [2\delta] - \left[\delta + \frac{2}{p^k}\right] \geqslant 1 - 1 = 0.$$

Therefore, for f_p to be negative, we must have both $0 \leqslant \delta < 1/2$ and $\delta + 2/p^k \geqslant 1$. These two inequalities force p^k to be less than 4, that is, p must be 2 or 3 and k must be 1. Thus each exponent f_p is nonnegative except for $p = 2$ and $p = 3$ when f_p can be -1. But this is exactly balanced by the 6 in $6C_n/(n+2)$.

J. That is an extremely ugly proof. You have treated the numbers as objects and disregarded the fact that they arose from concrete practical problems.

P. What is this noise? I got a proof, didn't I? You want me to deal in applied mathematics or pure mathematics?

J. I just thought I'd mention that since all of the Catalan numbers are integers, I can rewrite $C_n = 4C_{n-1} - (6C_{n-1}/(n+1))$ as $6C_{n-1}/(n+1) = 4C_{n-1} - C_n$ and this not only tells me that $(n+1)$ divides $6C_{n-1}$ integrally, it tells me the actual value of division. Isn't that a simple proof?

P. It may be simple, but I feel sullied, used, betrayed. Your proof comes from information about the origin of the Catalan numbers rather than an understanding of the Catalan numbers as objects of pure thought.

Calculating large Catalan numbers

P. I'd like to calculate more than the 64th Catalan number. I don't like to be restricted to integers with less than 37 digits. I want to calculate C_{60000}.

J. Why in the world do you want to calculate such a large number?

P. Because it's there. Now how do we do it?

J. We could calculate C_{60000} as a string of symbols rather than as a number.

P. What do you mean?

J. The computer can store arbitrarily long strings of symbols. By writing a symbolic multiplier and a symbolic adder, we can trick the computer into doing exact arithmetic on arbitrarily large integers. However, there is a price to pay. Instead of doing multiplication using hardware which is fast, the computer will have to do the multiplication using software which is slow. Furthermore, to multiply a number m digits long by a number n digits long in this fashion will take mn one-digit multiplications and mn one-digit additions.

It is unfortunate that your four previous representations of C_n have required division.

P. But there is a way to compute the Catalan numbers that doesn't involve division! The Catalan numbers satisfy the recurrence relation:

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0. \tag{5}$$

Thus, by saving the Catalan numbers C_0 through C_{n-1}, we can compute the nth Catalan number by n multiplications, n additions, and no divisions.

J. Unfortunately, this fifth approach takes a great deal of space and time.

P. You're being vague. How much time and how much space?

J. That, dear Peter, depends on n. We better first estimate the size of the intermediate calculations. Let's let m be an arbitrary integer between 1 and 60000. Since C_m equals $(2m)!/(m!(m+1)!)$, then by Stirling's estimate of $m! \sim \sqrt{2\pi m}\,(m/e)^m$ we obtain

$$C_m \sim \frac{\sqrt{4\pi m}\,(2m/e)^{2m}}{\sqrt{2\pi m}\,(m/e)^m\sqrt{2\pi(m+1)}\,((m+1)/e)^{m+1}} \sim \frac{2^{2m}}{\sqrt{\pi}\,m^{3/2}}.$$

This shows that the number of digits in C_m is proportional to m itself.

P. Aha. I can now estimate how long it will take to compute C_n from $C_0,\dots,C_{n-1}$. To compute C_n with this approach, we multiply C_j, a number with $O(j)$ digits, by C_{n-j}, a number with $O(n-j)$ digits, as j goes from zero to n, a total of

$$O\left(\sum_{j=0}^{n} j(n-j)\right) = O(n^3)$$

multiplications. In particular, using the fifth approach to calculate C_n from scratch involves $O(\sum_{k=1}^{n} k^3) = O(n^4)$ multiplications.

J. Well, Peter, I hope you can see that the computation of C_{60000} is prohibitive from the time standpoint of $O(n^4)$ operations. It is also prohibitive from the standpoint of space, since to compute C_{n+1} we would need to store the numbers $C_0, C_1,\dots,C_n$ simultaneously. Since C_j is $O(j)$ digits in length we would use $O(\sum_{j=0}^{n} j) = O(n^2)$ storage to compute C_n.

P. All is vanity. You have merely teased me about the power of these stupid machines. Asking for the 60000th Catalan number and you tell me I need $O((60000)^4)$ time and $O((60000)^2)$ space!

J. Don't blame me, you were the one who suggested the recurrence relation (5). The real question is: How can we calculate C_{60000} directly, without calculating $C_0, C_1,\dots,C_{59999}$? Why don't we use the computer to give itself instructions?

P. What do you mean? You don't mean let the machine think or self-modify do you?

J. No, no. I mean we let it do some preliminary calculations whose results we don't know and then let it use these intermediate results in a general strategy which we design.

P. Sounds great, but how does it apply to the Catalan number C_n, where n is a very large integer?

J. Since C_n is an integer given by

$$\frac{2n(2n-1)\cdots(n+3)(n+2)}{2\cdot 3\cdots n}, \tag{6}$$

each term in the denominator must integrally divide the numerator. We let the computer create a string for each of the $n-1$ symbols $2n$ through $n+2$ (the $n-1$ factors in the numerator). Then we let the computer find a string which 2 (the first factor in the denominator) divides integrally, and divide that string by 2. Then the division process is repeated for 3, the second factor in the denominator, and so on. When the computer has finished all this division, there will be $n-1$ strings, none representing a number larger than $2n$. The computer will then multiply the strings that are left to compute C_n.

P. You've proposed a scheme—but how do we know it's any better than the previous one? Can you give any time and space bounds?

J. Sure. First let's bound the number of required single digit multiplications. Recall that the number of digits in a number m is proportional to $\ln m$. Therefore, multiplying digit by digit two terms of the numerator, each of length at most $k\ln 2n$, takes at most $k^2(\ln 2n)^2$ single digit multiplications, and produces a number at most $2k\ln 2n$ digits long. It we take this number (at most $2k\ln 2n$ digits long) and multiply it by a term of the numerator (at most $k\ln 2n$ digits long), then it will take at most $2k^2(\ln 2n)^2$ single digit multiplications and will produce a number at most $3k\ln 2n$ digits long. Repeating this $n-2$ times will produce C_n after at most

$$\sum_{j=1}^{n-2} j\left(k^2(\ln 2n)^2\right) = O(n\ln n)^2 \tag{$*$}$$

single digit multiplications. A considerable improvement in time!

P. This sounds fine, except for two things. First, since you don't know which term of the denominator will divide which term of the numerator, you may have to perform $n-1$ divisions for each of the $n-1$ terms of the denominator. Thus, it appears you may have to do $(n-1)^{n-1}$ divisions just to write C_n as a product of $n-1$ numbers.

Second, although each term of the denominator does divide the numerator, it does not always divide a *single* term of the numerator after some divisions have taken place. For example, consider C_7. If you try to divide the terms from 2 to 7 in the denominator into terms from 14 to 9 in the numerator, you find:

$$\frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}=\frac{7\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{3\cdot 4\cdot 5\cdot 6\cdot 7}=\frac{7\cdot 13\cdot 4\cdot 11\cdot 10\cdot 9}{4\cdot 5\cdot 6\cdot 7}$$
$$=\frac{7\cdot 13\cdot 11\cdot 10\cdot 9}{5\cdot 6\cdot 7}=\frac{7\cdot 13\cdot 11\cdot 2\cdot 9}{6\cdot 7}$$

and the process breaks down. Even if you divide factors in the numerator by terms in the denominator in a different order, the same problem occurs. Thus, if a term of the denominator doesn't divide one of the $n-1$ terms of the numerator at some stage, you would have to try dividing it into one of the $(n-1)(n-2)$ products of pairs of terms of the numerator, and when this fails, try dividing it into one of the $(n-1)(n-2)(n-3)$ products of triples of terms of the numerator, etc.

J. Well, wipe that snide smile off your face. The bounds I propose look good; let's save the appearances.

P. There may be a way to salvage your formula (6). I suggest we return to the machinery of my "extremely ugly proof" that $6C_n/(n+1)$ is an integer. I propose to compute C_n by this scheme:

First factor $(2n)!$, $n!$ and $(n+1)!$ each as a product of primes; next, cancel the common prime factors in the numerator and denominator of $\dfrac{(2n)!}{n!(n+1)!}$, and finally, expand what is left as strings of digits. (7)

J. Are you out of your everliving gourd! You want to factor 120000!. Do you know how big 120000! is? By Stirling's estimate of $n!$, it is approximately 557000 digits long!

Surely you are aware of the difficulty of factoring a number. The basis of modern cryptographic techniques relies on the fact that to factor an arbitrary integer with a mere 200 digits using the fastest known algorithm at one operation per microsecond takes 3.8×10^9 years, and a number of 500 digits long takes 4.2×10^{25} years [**10**]. To factor an arbitrary number of 557000 digits would take, roughly, 10^{1800} years.

P. Where's your faith? Look, I didn't say we were factoring an *arbitrary* number 557000 digits long. I said let's factor 120000!.

J. Big wow. So it's a little bit more specific. So what?

P. Remember how I said that for any integer m,

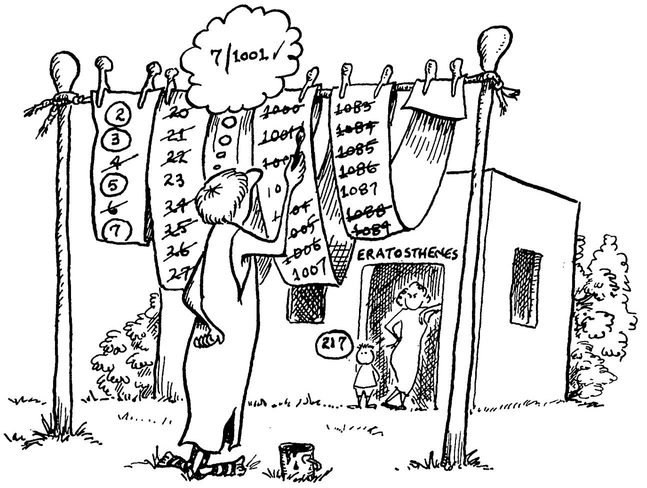
$$m! = \prod_p p^{e_p}, \text{ where } e_p = \sum_{k=1}^{\infty}\left[\frac{m}{p^k}\right].$$

J. Sure, I remember. It's still ugly.

P. But it is general, abstract, independent of any peculiarities of the Catalan numbers and that's what gives it power. Since $[m/p^k]$ is zero for $k > (\ln m)/\ln p$, we only have to find the primes up to 120000 and then compute e_p for these primes.

J. Well, in that case, I may have some good news for you. I have something for computing primes which is better than those programs which test for primality of the odd number n by dividing by 3 and then incrementing by 2 until the divisor is bigger than $\sqrt{n}$.

Here is a fast way of getting all primes less than 120000.




```
Mark off the first 120000 numbers (except for 1);
While all numbers aren't crossed off
  Begin
    Circle the first uncrossed number;
    Cross off all (additive) multiples of the number you just circled;
  End;
```

The circled numbers are the primes.

P. That is just the Sieve of Eratosthenes.

J. So? I don't care what it's called. Can you give any bounds on how long it takes to generate all the primes up to m using this method?

P. Sure. If m is an arbitrary positive integer, then the total number of operations to find all primes less than m can be bounded quite easily. Crossing off all multiples of 2 takes $m/2$ additions. Crossing off all multiples of 3 takes $m/3$ additions. Continuing we see that the entire algorithm requires $m\Sigma p^{-1}$ additions, where p runs through all primes less than or equal to m.

J. Peter, that's elegant but useless unless you can bound that sum Σp^{-1}.

P. Oh, but I can. However, it is nontrivial and will require Riemann-Stieltjes integration and the prime number theorem.

J. The prime number theorem? You mean the statement that the number of primes less than x is given approximately by $x/\ln x$?

P. Yes. Treating the summation as a Riemann-Stieltjes integral against $\pi(x)$, where $\pi(x)$ denotes the number of primes less than x, we can rewrite Σp^{-1} as

$$\sum_{p\leqslant m}\frac{1}{p}=\int_2^m\frac{d\pi(x)}{x}.$$

Integrating by parts yields

$$\int_2^x\frac{d\pi(x)}{x}=\frac{\pi(x)}{x}+\int_2^x\frac{\pi(x)\,dx}{x^2}.$$

Using the prime number theorem that $\pi(x)\sim x/\ln x$ we obtain

$$\int_2^x\frac{d\pi(x)}{x}\sim\frac{x}{x\ln x}+\int_2^x\frac{x\,dx}{x^2\ln x}=\frac{1}{\ln x}+\ln\ln x.$$

Thus $\Sigma_{p\leqslant m}p^{-1}$ is essentially $\ln\ln m$ and therefore it takes $m\ln\ln m$ additions to find all primes less than m. In particular, it only takes about 295000 additions (not multiplications!)

to find all primes less than 120000.

J. Great. But this still doesn't tell us how to factor 120000!.

P. True, but give me some time. The expression

$$e_p=\sum_{k=1}^{\infty}\left[\frac{m}{p^k}\right]$$

has only a finite number of nonzero terms. In fact, $[m/p^k]$ is zero as soon as p^k is bigger than m, which will be when $k>(\ln m)/\ln p$. Thus the total number of divisions to calculate e_p is $[(\ln m)/\ln p]$. The total number of divisions to calculate $n!$ is therefore bounded by:

$$\sum_{p\leqslant m}\frac{\ln m}{\ln p}=\ln m\sum_{p\leqslant m}\frac{1}{\ln p}.$$

J. I know, you're going to use the same techniques to bound $\Sigma 1/\ln p$ as you did for Σp^{-1}.

P. Correct! We can write

$$\sum_{p\leqslant m}\frac{1}{\ln p}=\int_2^x\frac{1}{\ln x}\,d\pi(x),$$

and integrate by parts and use the prime number theorem to get

$$\int_2^x\frac{1}{\ln x}\,d\pi(x)=\frac{\pi(x)}{\ln x}+\int_2^x\frac{\pi(x)\,dx}{x(\ln x)^2}\sim\frac{x}{(\ln x)^2}+\int_2^x\frac{dx}{(\ln x)^3}.$$

Letting $G(x)=x(\ln x)^{-2}$ and $F(x)=\int_2^x(\ln x)^{-3}\,dx$, we quickly can check that $G(e^3)>F(e^3)$ and $G'(x)>F'(x), x\geqslant e^3$. Thus $\int(\ln x)^{-1}\,d\pi(x)\sim x(\ln x)^{-2}$ and therefore, the total number of divisions to calculate $m!$ is

$$\sum_{p\leqslant m}\frac{\ln m}{\ln p}\sim\frac{m}{\ln m}.$$

J. Let me see if I understand your proposal. To factor $(2n)!$, you are going to take approximately $O(n\ln\ln n)$ additions and $O(n)$ space and generate all of the primes up to $2n$.

P. Correct. Then I'll take each prime and calculate for p the quantity:

$$e_p=\sum_{k=1}^{\infty}\left[\frac{2n}{p^k}\right].$$

It will take $[(\ln 2n)/\ln p]$ divisions to do each e_p and it'll take $O(n/\ln n)$ total divisions to calculate all the e_p.

J. And then you'll write the prime factorization of $(2n)!$ as

$$\prod p^{e_p}.$$

How incredibly sly. You're right. Factoring large factorials is not tough!

P. We can proceed to multiply these primes as strings to create C_n. Using your previous bounds we know this can be done in $O(n\ln n)^2$ single digit operations.

J. Thanks Peter, but my previous bounds are probably useless since they were derived from my erroneous assumption that each of the $n-1$ terms in the denominator divides a term of the numerator.

P. Do you mean, Jill, that all that was for nothing?

J. Hopefully not. But since you didn't give a bound on the number of primes that are actually in C_n we can't bound the number of multiplications that will be needed. Why don't we compute the prime decompositions of a couple of C_n and get some insight. After all, "the purpose of computation is insight, not numbers". Here are two factorizations.

$$C_{30}=2^4\cdot 7\cdot 11\cdot 17\cdot 19\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\cdot 59;$$

$$C_{173}=2^4\cdot 3\cdot 5^2\cdot 7^3\cdot 11\cdot 17\cdot 23\cdot 31\cdot 37\cdot 47\cdot 59\cdot 61\cdot 67\cdot 89\cdot 97\cdot 101\cdot 107\cdot 109$$
$$\cdot 113\cdot 179\cdot 181\cdot 191\cdot 193\cdot 197\cdot 199\cdot 211\cdot 223\cdot 227\cdot 229\cdot 233$$
$$\cdot 239\cdot 241\cdot 251\cdot 257\cdot 263\cdot 269\cdot 271\cdot 273\cdot 279\cdot 281\cdot 283\cdot 293\cdot 307$$
$$\cdot 311\cdot 313\cdot 317\cdot 331\cdot 337.$$

P. Hmm. Not all primes appear and if a prime appears it usually appears with exponent 1.

J. There is a simple explanation for this. You showed that the exponent of any prime p in C_n is given by

$$\sum_{k=1}^{\infty}\left[\frac{2n}{p^k}\right]-\left[\frac{n}{p^k}\right]-\left[\frac{n+1}{p^k}\right],$$

but you didn't point out that each difference

$$\left[\frac{2n}{p^k}\right]-\left[\frac{n}{p^k}\right]-\left[\frac{n+1}{p^k}\right]$$

is always 0 or 1, an observation which a computer printout of 10000 cases makes painfully obvious. It is easy to see why this must be true.

Since

$$p^k\left[\frac{2n}{p^k}\right]+a_1=2n, \qquad 0\leqslant a_1<p^k,$$
$$p^k\left[\frac{n}{p^k}\right]+a_2=n, \qquad 0\leqslant a_2<p^k,$$
$$p^k\left[\frac{n+1}{p^k}\right]+a_3=n+1, \qquad 0\leqslant a_3<p^k,$$

where each a_i is an integer, we have:

$$\left[\frac{2n}{p^k}\right]-\left[\frac{n}{p^k}\right]-\left[\frac{n+1}{p^k}\right]=\left(\frac{2n}{p^k}-\frac{a_1}{p^k}\right)-\left(\frac{n}{p^k}-\frac{a_2}{p^k}\right)-\left(\frac{n+1}{p^k}-\frac{a_3}{p^k}\right)=\frac{a_2+a_3-a_1-1}{p^k}.$$

But $\left[\frac{2n}{p^k}\right]-\left[\frac{n}{p^k}\right]-\left[\frac{n+1}{p^k}\right]$ is an integer, so we only need to prove that

$$-1<\frac{a_2+a_3-a_1-1}{p^k}<2$$

in order to conclude that it is always either 0 or 1.

Since $a_2+a_3-a_1-1<a_2+a_3<p^k+p^k$, it's clear that $(a_2+a_3-a_1-1)/p^k<2$. Since at least one of the integral terms a_2 and a_3 must be positive, $a_2+a_3-1\geqslant 0$; hence, $-p^k<-a_1\leqslant a_2+a_3-a_1-1$, so $-1<(a_2+a_3-a_1-1)/p^k$.

P. My congratulations. That is at least as ugly as anything I've produced today.

J. Ah. Jealousy ill becomes you Peter. What this shows is the number of times a prime p appears in the Catalan number C_n is bounded by the number of times

$$\left[\frac{2n}{p^k}\right]-\left[\frac{n}{p^k}\right]-\left[\frac{n+1}{p^k}\right]$$

is nonzero. Since the term is certainly 0 whenever $p^k>2n$, the number of times a prime p appears in the Catalan number C_n is bounded by $[(\ln 2n)/\ln p]$.

In particular, in the prime factorization of C_n, a prime between $2n$ and $(2n)^{1/2}$ appears at most once, a prime between $(2n)^{1/2}$ and $(2n)^{1/3}$ appears at most twice, etc.

Recall that my original estimate of $O(n\ln n)^2$ single digit multiplications to compute C_n

was based on having no more than n terms left in the numerator after doing the divisions. The factorization of C_{30} as 14 terms and C_{173} as 54 terms suggests that there are in fact far fewer than n terms in the prime factorization of C_n.

P. This should let us give both an upper and a lower estimate for the number of terms in the prime factorization of C_n. In fact, since C_n always involves every prime between $n+1$ and $2n$ (there is nothing in the denominator to cancel them) the number of terms in the prime factorization of C_n is bounded below by

$$(\pi(2n) - \pi(n+2)) \sim \left(\frac{2n}{\ln 2n} - \frac{n+2}{\ln(n+2)}\right) \sim \frac{n}{\ln n}.$$

On the other hand, each prime in the interval from $2n$ to $(2n)^{1/2}$ appears in C_n at most once. Each prime in the interval $(2n)^{1/2}$ to $(2n)^{1/3}$ appears in C_n at most twice. Continuing in this manner, I can see that the total number of terms in the prime factorization of C_n is bounded above by

$$\sum_{k=1}^{j} k\left[\pi\left((2n)^{1/k}\right) - \pi\left((2n)^{1/(k+1)}\right)\right],$$

where j is the maximum possible exponent of any prime in C_n.

J. But how are you going to bound that sum?

P. Fortunately this is a telescoping series. If you expand it, you get

$$\pi(2n) + \pi\left((2n)^{1/2}\right) + \pi\left((2n)^{1/3}\right) + \cdots + \pi\left((2n)^{1/j}\right) - j\pi\left((2n)^{1/(j+1)}\right).$$

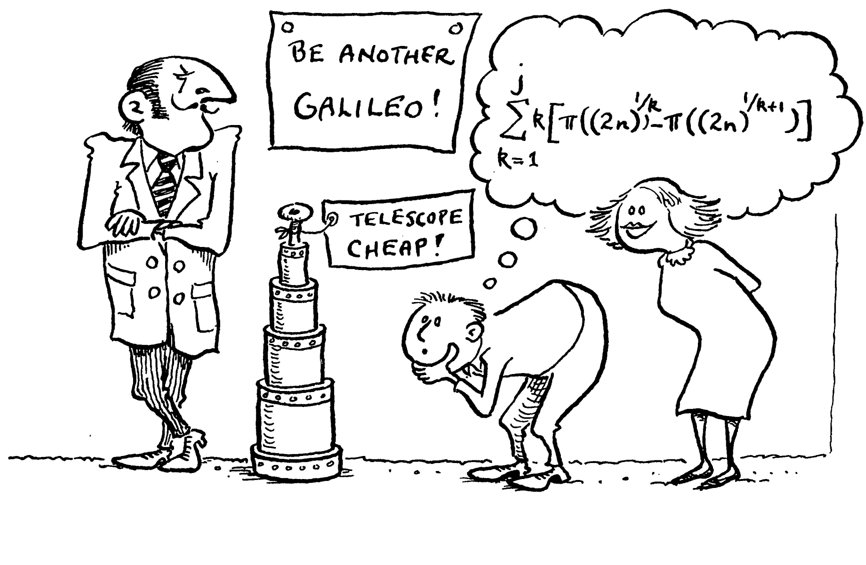
Since $\pi(x)$ is an increasing function of x and $(2n)^{1/j}$ is decreasing in j, we obtain the trivial bound

$$\pi(2n) + \pi\left((2n)^{1/2}\right) + \cdots + \pi\left((2n)^{1/j}\right) < \pi(2n) + (j-1)\pi\left((2n)^{1/2}\right).$$

But j, the maximum possible exponent of any prime in C_n, is bounded by $(\ln 2n)/\ln 2$. Consequently, the maximum number of factors is bounded above by

$$\pi(2n) + (j-1)\pi\left((2n)^{1/2}\right) \sim \left\{\frac{2n}{\ln 2n} + \frac{\ln 2n}{\ln 2}\frac{(2n)^{1/2}}{\ln\left((2n)^{1/2}\right)}\right\} \sim \frac{2n}{\ln n}.$$

Therefore the true number of terms (counting multiplicities) in the prime factorization of C_n is trapped between $n/\ln n$ and $2n/\ln n$.



J. That's it then! Now that we have upper and lower bounds on the number of primes to expect in the prime representation of C_n we can give an upper and lower bound for the number of single digit multiplications needed to compute C_n using this representation. Since the $n/\ln n$ primes between $n+2$ and $2n$ are each factors of C_n, and since each has $O(\ln n)$ digits, then the same counting technique we used earlier $(*)$ shows there will be at least $O(n^2)$ single-digit operations. On the other hand, since there are at most $2n/\ln n$ factors in C_n, each of length at most $O(\ln 2n)$, then the same analysis shows there will be at most $O(n^2)$ single digit operations.

P. That means my method of prime factorization will require essentially n^2 single digit multiplications.

J. And, best of all, that's a better bound than my $n^2(\ln n)^2$ bound derived under my simpler but erroneous scheme of dividing the denominators one by one into the numerator!

P. Let me go over this one more time. Procedure (7) lets us compute C_n for large values of n by first computing the necessary primes in $O(n \ln\ln n)$ additions, then computing the exponents of the prime factorization in $O(n/\ln n)$ divisions, and finally by multiplying the roughly $O(n/\ln n)$ remaining strings in at most $O(n^2)$ single digit operations.

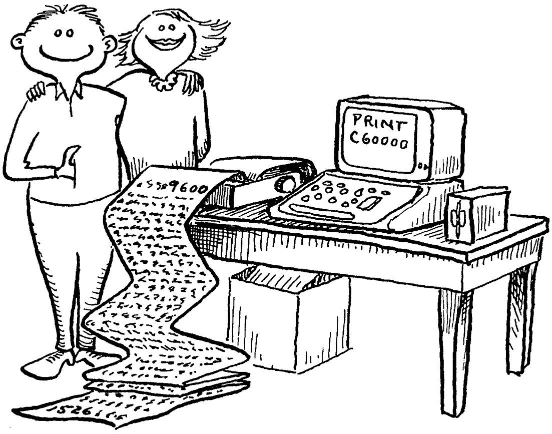
J. By the way, the storage of the primes less than $2n$ does not require excessive space. In fact, since each prime p is $O(\ln p)$ digits in length, the total space for all primes less than $2n$ is

$$\sum_{p \leqslant 2n} \ln p = \int_2^{2n} \ln x \, d\pi(x) = \pi(x)\ln x - \int_2^{2n} \frac{\pi(x)}{x}\, dx$$

using the techniques used before. Therefore $\sum_{p \leqslant 2n} \ln p$ is bounded by $\pi(2n)\ln(2n)$, which is $O(n)$.

Postscript

The Catalan numbers, as so many other numbers in applied mathematics, can be viewed as abstract ideas, devoid of physical interpretation. The actual computation of the numbers can depend on how the number is represented. The determination of large Catalan numbers illustrates time and space limitations, the heart of computer science. The determination of bounds on time and space requirements for the computation of the Catalan numbers returns the computer science student repeatedly to properties of primes, the prime number theorem, and Stieltjes integration. The final result was a procedure for computing the nth Catalan number with $O(n^2)$ single digit multiplications in $O(n)$ space. For those interested in software that allows the manipulation of large integers, we mention MuMath, a symbolic algebra program available from Microsoft, which directly supports integers up to several tens of thousands of digits. It is available for the TRS-80, the Apple II and the IBM PC. Using this very useful program, one finds that C_{60000} begins with 1526... and ends with... 9600 and has 36109 intermediate digits.



References

[1] H. W. Becker, Problem 4277 (solution), Amer. Math. Monthly, 56 (1949) 697–699.
[2] W. G. Brown, Historical note on a recurrent combinatorial problem, Amer. Math. Monthly, 72 (1965) 973–977.
[3] N. G. de Bruijn and B. J. M. Morselt, A note on plane trees, J. of Comb. Theory, 2 (1967) 27–34.
[4] W. Feller, An Introduction to Probability Theory and its Applications, John Wiley & Sons, New York, 1950.
[5] H. G. Forder, Some Problems in Combinatorics, Math. Gazette, 45 (1961) 199–201.
[6] Martin Gardner, Mathematical Games, Catalan numbers: an integer sequence that materializes in unexpected places, Scientific American, vol. 234, no. 6 (June, 1976) 120–125.
[7] D. A. Klarner, Correspondences between plane trees and binary sequences, J. of Comb. Theory, 9 (1970) 401–411.
[8] D. E. Knuth, The Art of Computer Programming, Vol. 1, Addison-Wesley, New York, 1973.
[9] E. Lucas, Théorie des Nombres, Paris, 1891.
[10] R. L. Rivest, A. Shemir, L. Adleman, A method for obtaining digital signatures and public key cryptosystems, Comm. ACM, 21 (1978) 120–126.
[11] G. Szekeres, Problem 3954, Amer. Math. Monthly, 48 (1941) 565.
[12] Jacobus H. vanLint, Combinatorial Theory Seminar Eindhoven University of Technology, Lecture Notes in Mathematics, v. 382, Springer-Verlag, 1974.

Conversation Piece

A physicist told me
The universe is pear-shaped.
I said, Then what's out there
Beyond the pear?
He laughed, and tried to set me straight.
But either he was too acute
Or I was too obtuse.

I told the physicist
About Gödel's undecidability.
He listened, but I lost him.
He said, Aren't you straying into the domain
Of Metaphysics?
Perhaps, I said, But it's in the range of
Metamathematics.

—KATHARINE O'BRIEN

References

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—KATHARINE O'BRIEN

Stirling Ideas for Freshman Calculus

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At the mention of Stirling's formula

$$n! \sim \sqrt{2\pi}\sqrt{n}\left(\frac{n}{e}\right)^n$$

most mathematicians automatically think of the integral representation

$$n! = \int_0^\infty t^n e^{-t}\,dt$$

and a derivation by the method of steepest descent. A consequence of this reflex is that neither the formula nor anything resembling it appears as standard fare in first-year calculus courses. This is unfortunate because Stirling's formula is needed in a great many second year courses—probability, statistics, combinatorics and such courses in physics and chemistry as touch on statistical mechanics.

The purpose of this paper is to suggest some well-motivated and appealing exercises based on core material from first year calculus that yield "junior versions" of Stirling's formula of the form

$$k_1\sqrt{n}\left(\frac{n}{e}\right)^n \leqslant n! \leqslant k_2\sqrt{n}\left(\frac{n}{e}\right)^n. \tag{1}$$

These fit nicely into first-year calculus courses where they serve as an interesting application of standard techniques. They also make a worthwhile addition to later courses in probability where the chief interest is to apply Stirling's formula. In these courses, the proof and preliminary use of a junior version not only fills the plausibility gap which results when a mysterious cookbook formula is suddenly announced, but also contributes useful insight into the nature of the approximation involved when the real Stirling's formula is brought into play.

Here is a demanding test example to show how well this approach can serve. We consider a particle which sets out from the origin on a one-dimensional random walk. The particle takes successive steps of $+1$ or -1 with equal likelihood and we seek the probability of its return to the origin after m steps. Since a return to the origin entails an equal number of $+1$ steps and -1 steps, the required probability is

$$p(m) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \dfrac{(2n)!}{(n!)^2 2^{2n}} & \text{if } m = 2n \text{ is even.} \end{cases}$$

It follows from (1) that

$$\frac{k_1\sqrt{2n}\left(\dfrac{2n}{e}\right)^{2n}}{k_2^2 n\left(\dfrac{n}{e}\right)^{2n} 2^{2n}} \leqslant p(2n) \leqslant \frac{k_2\sqrt{2n}\left(\dfrac{2n}{e}\right)^{2n}}{k_1^2 n\left(\dfrac{n}{e}\right)^{2n} 2^{2n}},$$

hence

$$\frac{2k_1}{k_2^2}\frac{1}{\sqrt{2n}} \leqslant p(2n) \leqslant \frac{2k_2}{k_1^2}\frac{1}{\sqrt{2n}}.$$

Here the $1/\sqrt{m}$ behavior emerges and we can justify the exact asymptotic expression $p(2n) \sim 1/\sqrt{\pi n}$ by stating (without proof) that k_1 and k_2 can be taken arbitrarily close to $\sqrt{2\pi}$ if m is sufficiently large.

As a warmup for our later work, here is a very fast estimate for $n!$. From the Taylor series for the exponential function we see

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + \cdots > \frac{x^n}{n!}, \qquad \text{if } x > 0;$$

if we put $x = n$, we obtain

$$n! > \left(\frac{n}{e}\right)^n.$$

For an upper bound, we recall that the product of n positive numbers with prescribed sum is maximized when the numbers are equal. Since

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

is the product of n numbers whose sum is $n(n+1)/2$, we have

$$n! < \left(\frac{n+1}{2}\right)^n.$$

The estimate

$$\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+1}{2}\right)^n \qquad (2)$$

gives a rough idea of how quickly $n!$ grows but it is too weak to be useful in our random walk problem (we did say this is a demanding test case) since it yields only

$$\frac{1}{e^2}\left(\frac{2}{e}\right)^{2n} < p(2n) < e\left(\frac{e}{2}\right)^{2n}.$$

The upper bound tends to infinity with n so it is less effective than the trivial bound $p(2n) \leqslant 1$ which follows because $p(2n)$ is a probability. The lower bound decreases to zero as a geometric sequence and gives no hint that the actual behavior is like $1/\sqrt{\pi n}$.

Before going on to better but slightly more complicated estimates of $n!$ we add one more quick result. If a sequence tends to a limit, then the new sequence whose nth term is the average or the geometric mean of the first n terms of the original sequence tends to the same limit. Since

$$\left(1 + \frac{1}{n}\right)^n \to e,$$

it follows that

$$\left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$$

and hence, if we take geometric means,

$$\sqrt[n]{\left(\frac{1}{2}\right)^1\left(\frac{2}{3}\right)^2\left(\frac{3}{4}\right)^3 \cdots \left(\frac{n}{n+1}\right)^n} \to \frac{1}{e}.$$

But this simplifies to

$$\frac{\sqrt[n]{n!}}{n+1} \to \frac{1}{e},$$

which is equivalent to

$$\frac{\sqrt[n]{n!}}{n} \to \frac{1}{e}$$

and indicates that $n!$ is closer to $(n/e)^n$ than to $((n+1)/2)^n$.

The main theorem

The version of Stirling's formula we prove next is

THEOREM 1. *For* $n = 1, 2, 3, \ldots$

$$\frac{e}{\sqrt{2}}\sqrt{n}\left(\frac{n}{e}\right)^n \leqslant n! \leqslant \frac{e^2}{2}\sqrt{n}\left(\frac{n}{e}\right)^n.$$

In the notation of (1), this theorem says that the constants k_1 and k_2 can be taken as

$$k_1 = \frac{e}{\sqrt{2}} \approx 1.922 \quad \text{and} \quad k_2 = \frac{e^2}{2} \approx 3.695.$$

We note for comparison with Stirling's formula, that

$$\sqrt{2\pi} \approx 2.507.$$

The proof of Theorem 1 is based on various estimates of the sum

$$\ln n! = \ln 2 + \ln 3 + \cdots + \ln n \tag{3}$$

by integrals. In the simplest version of this method, we have

$$\int_1^n \ln t\, dt < \ln n! < \int_2^{n+1} \ln t\, dt$$

or

$$n \ln n - n + 1 < \ln n! < (n+1)\ln(n+1) - (n+1) + 2 - 2\ln 2,$$

which yields

$$e\left(\frac{n}{e}\right)^n < n! < \frac{e^2}{4}\left(\frac{n+1}{e}\right)^{n+1}.$$

The lower bound is essentially what we had in (2), but the upper bound is a considerable improvement on $((n+1)/2)^n$. Nevertheless, this pair of bounds is not sharp enough to be useful in our random walk problem. It yields

$$\frac{16}{e^3}\left(\frac{1}{n+1}\right)^2 < p(2n) < \frac{2n+1}{2},$$

and once again the upper bound fails to show $p(2n) \leqslant 1$ and the lower bound is substantially smaller than the correct order of magnitude. In contrast, we have seen that bounds like those provided by Theorem 1 are effective in estimating this probability. We proceed to obtain these bounds.

To obtain the lower bound of the Theorem we note that the sum in (3) is equal to the area of a collection of rectangles which extend above the curve $y = \ln t$, $1 \leqslant t \leqslant n$, as shown in FIGURE 1. The tangent to $y = \ln t$ at $t = m$ cuts a triangle of area $1/2 \cdot 1/m$ off that portion of the rectangle of area $\ln m$ which extends above the curve. Thus

$$\ln n! > \int_1^n \ln t\, dt + \sum_{j=2}^{n} \frac{1}{2}\frac{1}{j}.$$

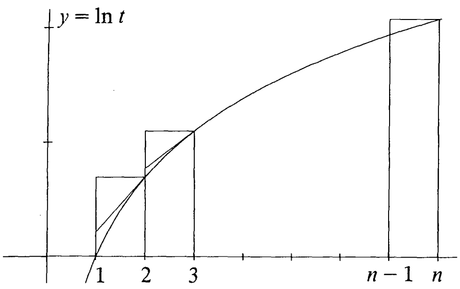

FIGURE 1. **A lower bound for $n!$ using the graph of $y = \ln t$, $1 \leqslant t \leqslant n$.**

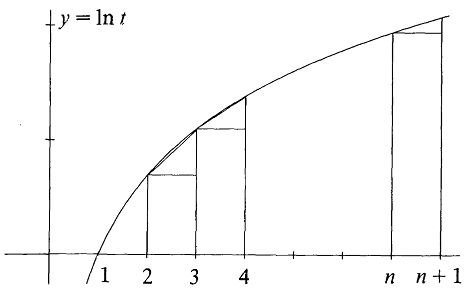

FIGURE 2. **An upper bound for $n!$ using the graph of $y = \ln t$, $2 \leqslant t \leqslant n + 1$.**

Estimating the second sum by a smaller integral, we obtain

$$\begin{aligned}\ln n! &> \int_1^n \ln t\,dt + \frac{1}{2}\int_2^{n+1}\frac{1}{t}\,dt\\ &= n\ln n - n + 1 + \frac{1}{2}\ln\frac{n+1}{2},\end{aligned}$$

which implies

$$n! > e\left(\frac{n}{e}\right)^n\sqrt{\frac{n+1}{2}} = \frac{e}{\sqrt{2}}\sqrt{n}\left(\frac{n}{e}\right)^n\sqrt{1+\frac{1}{n}} > \frac{e}{\sqrt{2}}\sqrt{n}\left(\frac{n}{e}\right)^n,$$

as required in the theorem.

To obtain the comparison upper bound we note that the sum in (3) is also equal to the area of a collection of rectangles which lie below the curve $y = \ln t$, $2 \leqslant t \leqslant n + 1$, as shown in FIGURE 2. Part of the region between this curve and the rectangle of area $\ln m$ is filled by a triangle of area $\frac{1}{2}[\ln(m+1) - \ln m]$. Thus

$$\begin{aligned}\ln n! &< \int_2^{n+1}\ln t\,dt - \sum_{j=2}^{n}\frac{1}{2}[\ln(j+1) - \ln j]\\ &= (n+1)\ln(n+1) - (n+1) + 2 - 2\ln 2 - \frac{1}{2}\ln\frac{n+1}{2},\end{aligned}$$

which implies

$$n! < \frac{e^2}{4}\left(\frac{n+1}{e}\right)^{n+1}\sqrt{\frac{2}{n+1}} = \frac{e^2}{2}\sqrt{n}\left(\frac{n}{e}\right)^n\left(\frac{n+1}{n}\right)^n\sqrt{1+\frac{1}{n}}\,\frac{\sqrt{2}}{2e} < \frac{e^2}{2}\sqrt{n}\left(\frac{n}{e}\right)^n.$$

This completes the proof of Theorem 1.

In simplifying the upper bound we replaced $((n+1)/n)^n$ by e and $\sqrt{1+1/n}$ by $\sqrt{2}$. The first replacement is justified by the easy lemma that $(1+1/n)^n$ increases to e but the second replacement is wasteful and shows that, for large n, the upper bound should hold with the constant $e^2/2$ replaced by $e^2/2\sqrt{2} \approx 2.612$, which is just slightly less than $e \approx 2.718$.

Improved estimates

We now prove by induction that $e^2/2$ can indeed be replaced by e in the upper bound of Theorem 1, i.e.,

$$n! \leqslant e\sqrt{n}\left(\frac{n}{e}\right)^n.$$

The case $n = 1$ starts the induction and also shows that the constant is best possible if the inequality is to hold for *all* $n \geqslant 1$. The inequality for $(n+1)!$ will follow from the inequality for $n!$

if it can be shown that

$$e\sqrt{n}\left(\frac{n}{e}\right)^n (n+1) < e\sqrt{n+1}\left(\frac{n+1}{e}\right)^{n+1}.$$

But this is equivalent to the inequality

$$e < \left(1+\frac{1}{n}\right)^{n+1/2} \tag{4}$$

which follows because the expression on the right in (4) decreases monotonically to e (see **[1]**).

Inspired by this easy success, we naturally attempt to obtain a lower bound

$$n! \geqslant k\sqrt{n}\left(\frac{n}{e}\right)^n$$

in the same way. Here the inequality for $(n+1)!$ would follow from the inequality for $n!$ if it could be shown that

$$k\sqrt{n}\left(\frac{n}{e}\right)^n (n+1) \geqslant k\sqrt{n+1}\left(\frac{n+1}{e}\right)^{n+1}$$

But this would contradict inequality (4).

There is, however, a curious "backwards induction" result that will prove useful to us: *If the inequality*

$$n! \geqslant k\sqrt{n}\left(\frac{n}{e}\right)^n$$

holds for one value of n, *then it holds for all smaller values.* This is true because the original inequality implies

$$(n-1)! \geqslant k\sqrt{n}\left(\frac{n}{e}\right)^{n-1}\frac{1}{e};$$

hence the result will hold provided

$$k\sqrt{n}\left(\frac{n}{e}\right)^{n-1}\frac{1}{e} \geqslant k\sqrt{n-1}\left(\frac{n-1}{e}\right)^{n-1},$$

or, equivalently, provided

$$\left(\frac{n}{n-1}\right)^{n-1/2} \geqslant e.$$

But this inequality does hold because it is just inequality (4) with n replaced by $n-1$.

Now we combine several techniques to prove a result which shows how to choose the constants k_1 and k_2 of inequality (1) arbitrarily close together.

If $n > m$, then, using the method which yields the lower bound in Theorem 1, we obtain

$$\begin{aligned}\ln n! &= \ln m! + \ln(m+1) + \ln(m+2) + \cdots + \ln(n)\\ &> \ln m! + \int_m^n \ln t\,dt + \frac{1}{2}\int_{m+1}^{n+1}\frac{1}{t}\,dt\\ &= \ln m! + (n\ln n - n) - (m\ln m - m) + \frac{1}{2}\ln\frac{n+1}{m+1},\end{aligned}$$

which implies

$$n! > \left[\frac{m!}{\sqrt{m+1}\left(\frac{m}{e}\right)^m}\right]\sqrt{n+1}\left(\frac{n}{e}\right)^n > \left[\frac{m!}{\sqrt{m+1}\left(\frac{m}{e}\right)^m}\right]\sqrt{n}\left(\frac{n}{e}\right)^n.$$

Our backwards induction then shows that

$$n! > \left[\frac{m!}{\sqrt{m+1}\left(\frac{m}{e}\right)^m}\right]\sqrt{n}\left(\frac{n}{e}\right)^n$$

holds for $n \leqslant m$. By combining these results, we establish

THEOREM 2. *For all m and n,*

$$\frac{m!}{\sqrt{m+1}\left(\frac{m}{e}\right)^m} < \frac{n!}{\sqrt{n}\left(\frac{n}{e}\right)^n}.$$

By applying the inequality of Theorem 2 once as it stands and once with the roles of m and n reversed, we obtain

COROLLARY 1. *For all m and n,*

$$\left[\frac{m!}{\sqrt{m+1}\left(\frac{m}{e}\right)^m}\right]\sqrt{n}\left(\frac{n}{e}\right)^n < n! < \left[\frac{m!}{\sqrt{m}\left(\frac{m}{e}\right)^m}\right]\sqrt{n+1}\left(\frac{n}{e}\right)^n.$$

Since the right hand inequality in Corollary 1 can be rewritten

$$n! < \left[\frac{m!\sqrt{1+\frac{1}{m}}\sqrt{1+\frac{1}{n}}}{\sqrt{m+1}\left(\frac{m}{e}\right)^m}\right]\sqrt{n}\left(\frac{n}{e}\right)^n$$

we also obtain

COROLLARY 2. *For all m and for all $n \geqslant m$,*

$$\left[\frac{m!}{\sqrt{m+1}\left(\frac{m}{e}\right)^m}\right]\sqrt{n}\left(\frac{n}{e}\right)^n < n! < \left[\frac{m!\left(1+\frac{1}{m}\right)}{\sqrt{m+1}\left(\frac{m}{e}\right)^m}\right]\sqrt{n}\left(\frac{n}{e}\right)^n.$$

Corollary 2 shows that the constants $k_1 = k_1(m)$ and $k_2 = (1+1/m)k_1$ can be chosen arbitrarily close together. For example, if $m = 10$, we have, for all $n \geqslant 10$,

$$2.409\sqrt{n}\left(\frac{n}{e}\right)^n < n! < 2.651\sqrt{n}\left(\frac{n}{e}\right)^n.$$

In closing, I remark that A. J. Coleman [2], H. Robbins [4], and Serge Lang [3] have also written about Stirling's formula from an elementary point of view. However, their methods are not quite as easy to motivate as those of the present treatment and therefore their work constitutes a logical sequel to this account.

My warm thanks are due to S. G. Whittington, J. B. Friedlander, P. J. Leah and the referee for contributing helpful suggestions.

References

[1] Frank Burk, Approximations to e, this MAGAZINE, 56 (1983) 34–36.
[2] A. J. Coleman, A simple proof of Stirling's formula, Amer. Math. Monthly 58 (1951) 334–336.
[3] Serge Lang, A First Course in Calculus, 4th edition, Addison-Wesley, 1978.
[4] H. Robbins, A remark on Stirling's formula, Amer. Math. Monthly 62 (1955) 26–29.

An Infinite Series for π with Determinants

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There are a variety of expressions for π. I have discovered an unusual one using determinants.

THEOREM.

$$\frac{\pi}{2}-1=2^0\begin{vmatrix}1 & 0 & 1\\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 1\\ 0 & 1 & 1\end{vmatrix}+2^1\begin{vmatrix}1 & 0 & 1\\ \frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{1}{2}\sqrt{2-\sqrt{2}} & 1\\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 1\end{vmatrix}$$

$$+2^2\begin{vmatrix}1 & 0 & 1\\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}} & 1\\ \frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{1}{2}\sqrt{2-\sqrt{2}} & 1\end{vmatrix}+\cdots$$

$$+2^{n-1}\begin{vmatrix}1 & 0 & 1\\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}_n}}} & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}_n}}} & 1\\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}_{n-1}}}} & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}_{n-1}}}} & 1\end{vmatrix}$$

$$+\cdots.$$

This can be proved with some geometry and calculus. A quarter of the area of a circle of unit radius can be expressed as the sum of an infinite series:

$$\frac{\pi}{4}=\Delta_0+2^0\Delta_1+2^1\Delta_2+2^2\Delta_3+\cdots+2^{n-1}\Delta_n+\cdots,$$

where the Δ_i $(i=0,1,2,\dots)$ are the areas of the isosceles triangles indicated in FIGURE 1. Let Q and P_0 be the points $(1,0)$ and $(0,1)$, respectively, and P_n be the apex of the triangle of area Δ_n $(n\geqslant 1)$.

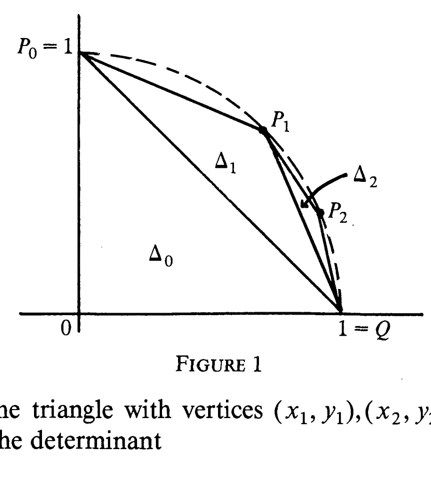

FIGURE 1

Recall that the area of the triangle with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ listed in counter-clockwise order is given by the determinant

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Since the nth triangle has vertices Q, P_n, and P_{n-1}, it is sufficient to identify the coordinates of each P_n. For each $n \geqslant 1$, P_n is the midpoint of the circular arc from P_{n-1} to Q. Hence, the tangent to the circle through P_n is parallel to the secant $P_{n-1}Q$. Rewriting this fact in analytic terms will provide the information we seek.

Let the coordinates of P_n be (u_n, v_n). The equation of the quarter circle is $y = f(x)$ $(0 \leqslant x \leqslant 1)$, where $f(x) = \sqrt{1-x^2}$. Since

$$f'(u_n) = \frac{f(u_{n-1}) - f(1)}{u_{n-1} - 1} \qquad (n \geqslant 1),$$

we find that

$$\frac{u_n}{\sqrt{1-u_n^2}} = \frac{\sqrt{1-u_{n-1}^2}}{u_{n-1}-1}.$$

Squaring both sides and solving for u_n yields $u_n^2 = \frac{1}{2}(1 + u_{n-1})$, which, with $u_n^2 + v_n^2 = 1$, produces $v_n^2 = \frac{1}{2}(1 - u_{n-1})$. Since u_n and v_n are both nonnegative and $u_0 = 0$, we obtain in succession

$$\begin{aligned}
(u_0, v_0) &= (0,1) \\
(u_1, v_1) &= \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \\
(u_2, v_2) &= \left(\sqrt{\frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)}, \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)}\right) = \left(\frac{1}{2}\sqrt{2+\sqrt{2}}, \frac{1}{2}\sqrt{2-\sqrt{2}}\right) \\
(u_3, v_3) &= \left(\sqrt{\frac{1}{2}\left(1+\frac{1}{2}\sqrt{2+\sqrt{2}}\right)}, \sqrt{\frac{1}{2}\left(1-\frac{1}{2}\sqrt{2+\sqrt{2}}\right)}\right) \\
&= \left(\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}, \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}\right) \\
&\vdots \\
(u_n, v_n) &= \left(\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}, \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}\right).
\end{aligned}$$

Thus the area of the nth triangle is

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ u_n & v_n & 1 \\ u_{n-1} & v_{n-1} & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0. & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\cdots\sqrt{2}}}}_n & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{\cdots\sqrt{2}}}}_n & 1 \\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\cdots\sqrt{2}}}}_{n-1} & \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{\cdots\sqrt{2}}}}_{n-1} & 1 \end{vmatrix},$$

from which the desired result is obtained.

I wish to thank Associate Editor Edward J. Barbeau for his helpful suggestions in revising the manuscript. This paper is dedicated to Mary K. Geffert.

References

[1] Petr Beckman, A History of π, St. Martin's Press, New York, 1971.
[2] Y. Mikami, The Development of Mathematics in China and Japan, Chelsea, New York, 1961, pp. 200–203, 213–217.

Intermediate Time Behavior of a Heat Equation Problem

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The way in which the solution of a heat equation problem approaches the steady-state temperature distribution can provide further insight into the problem and a better understanding of heat flow. Consider the following standard problem: A straight bar of homogeneous material of finite length l has a uniform cross section. We shall assume the sides are perfectly insulated and the cross-sectional dimensions are so small that the temperature is constant on any given cross section. We also assume the bar is in standard position with the x-axis along the axis of the bar and the ends at $x=0$ and $x=l$. Let $u=u(x,t)$ be the temperature of the bar where x is position and t is time, and let α^2 be the thermal diffusivity constant of the bar. The temperature of the bar is determined by the heat conduction equation $\alpha^2 u_{xx}=u_t$ for $0<x<l$, $t>0$, and the side conditions. The side conditions we shall consider are the initial condition $u(x,0)=b$ for $0<x<l$ and the boundary conditions $u(0,t)=0$ and $u(l,t)=a$ for $t\geqslant 0$. In other words, the bar is heated to a uniform temperature b; at time $t=0$ the left end is changed to a temperature of 0 while the right end is changed to temperature a, and these end temperatures are then maintained. To simplify the discussion we shall assume $a>0$ and $0<b\leqslant a/2$. The solution to this problem may be found in many texts such as [**1**] or [**2**].

Clearly the steady-state temperature distribution is $\lim_{t\to\infty}u(x,t)=(a/l)x$. If the temperature of the bar is plotted as a function of x for different times t, then a reasonable guess as to the intermediate time behavior of the solution is shown in FIGURE 1. Surprisingly, this behavior only occurs in the special case where $b=a/2$. If $0<b<a/2$, then for any x where $0<x<bl/a$ (where the initial temperature is greater than the steady-state temperature), substitution of large values of t into the solution for that x results in temperatures lower than the steady-state value. For $0<b<a/2$, even where the initial temperature is greater than steady-state, the temperature of the bar will converge to the steady-state value from below.

To verify and understand this behavior we must examine the solution of the problem. The solution has the form

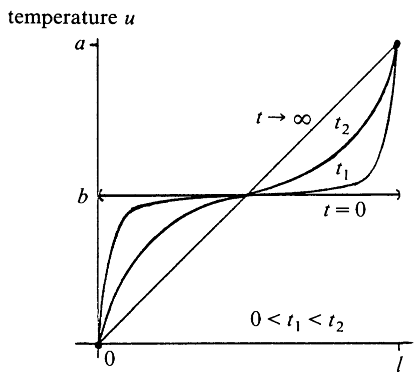

FIGURE 1

$$u(x,t)=\frac{a}{l}x+\sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/l^2)\sin(n\pi x/l)$$

where

$$c_n=\frac{2}{l}\int_0^l\left(b-\frac{a}{l}x\right)\sin(n\pi x/l)\,dx.$$

Elementary integration yields $c_n=2(2b-a)/n\pi$ if n is odd and $c_n=2a/n\pi$ if n is even. The solution can thus be arranged and written as

$$u(x,t)=\frac{a}{l}x+\sum_{n=1}^{\infty}\frac{a}{n\pi}\exp\left(-(2n)^2\pi^2\alpha^2 t/l^2\right)\sin(2n\pi x/l)$$
$$+\sum_{n=1}^{\infty}\frac{2(2b-a)}{(2n-1)\pi}\exp\left(-(2n-1)^2\pi^2\alpha^2 t/l^2\right)\sin((2n-1)\pi x/l). \qquad (1)$$

If $b<a/2$ then the dominant term in $u-(a/l)x$ as $t\to\infty$ is the first term of the second summation. The first summation has a common factor of $(a/\pi)\exp(-4\pi^2\alpha^2 t/l^2)$; removing this, the resulting series is dominated by a geometric series for $t>0$ and so the absolute value of the first summation is no greater than

$$(a/\pi)\exp(-4\pi^2\alpha^2 t/l^2)\left(1-\exp(-\pi^2\alpha^2 t/l^2)\right)^{-1}.$$

Similarly the second summation, minus the first term, is no greater in absolute value than the absolute value of

$$(2(2b-a)/\pi)\exp(-9\pi^2\alpha^2 t/l^2)\left(1-\exp(-\pi^2\alpha^2 t/l^2)\right)^{-1}.$$

For any x, $0<x<l$, if t is large then both these terms are negligible in comparison with

$$(2(2b-a)/\pi)\exp(-\pi^2\alpha^2 t/l^2)\sin(\pi x/l).$$

Hence this term does dominate as $t\to\infty$, and every factor of this term is positive with the exception of $2b-a$. Thus if $b<a/2$ then $u\to(a/l)x$ from below for any x, $0<x<l$, exactly as claimed.

To explain this behavior it is necessary to split up u. Let u_1 be the first two parts of equation (1) for u and let u_2 be the last summation. The first term, u_1, is easy to identify particularly if we observe that $u_2=0$ when $b=a/2$; u_1 is the solution to the heat conduction equation with initial condition $u_1(x,0)=a/2$ and boundary conditions $u_1(0,t)=0$ and $u_1(l,t)=a$. This is just the original problem in the special case where the initial temperature of the bar is exactly halfway between the endpoint temperatures. The second term, u_2, also turns out to be familiar; u_2 is the solution to the heat equation with initial condition $u_2(x,0)=b-a/2$ and boundary conditions $u_2(0,t)=0=u_2(l,t)$. In other words, u_2 is the solution to the problem in which a bar is cooled to a uniform temperature of $b-a/2$, and at time $t=0$ both ends are raised to a temperature of 0 and maintained there. A picture of the intermediate time behavior of the solution to this problem is shown in FIGURE 2.

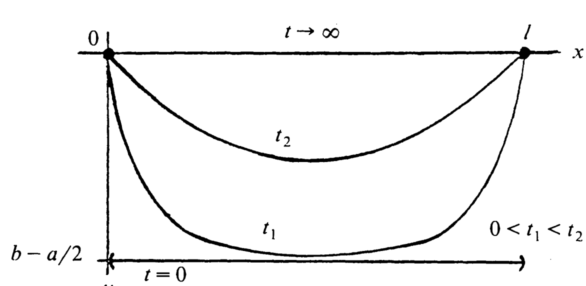

FIGURE 2

The two problems resulting from this decomposition of u are of very different types. The first is a temperature redistribution problem. With the initial temperature at $a/2$, the initial heat content of the bar is the same as that of the steady-state distribution and indeed, from symmetry considerations, is the same as that of any intermediate distribution. Of course there is heat flow since the end temperatures differ, but there is no change in the total heat content of the bar from this process. The only change induced is in the distribution of the temperature. The second problem, on the other hand, is a heat change problem. The result of this process is to raise the heat content of the bar to the level determined by the steady-state distribution.

Thus our original problem is comprised of two parts, one which raises the average temperature of the bar to the average temperature of the steady-state distribution and one which transforms a constant average temperature to the actual steady-state pattern. The solution to our problem is the result of superposition of the solutions of these attendant problems. Consideration of the exponential factors of the terms of these solutions explains the nature of the steady-state convergence noted earlier. Change in the heat content of the bar will always proceed more slowly than redistribution of the temperature. Consequently the temperature of the bar will approach steady-state from below, even when it is initially higher.

A numerical example is useful in illustrating these ideas. Suppose the bar is aluminum, in which case a reasonable value for α^2 is 0.86 cm^2/sec. Let $l = 20$ cm, $a = 60\,^\circ$C, and $b = 25\,^\circ$C. For $x = 5$ cm the temperatures at selected times are shown in TABLE 1.

t	$u_1(5, t)$	$u_2(5, t)$	$u(5, t)$
0	30.0	-5.0	25.0
5	27.354	-4.559	22.795
10	23.170	-3.859	19.311
20	18.498	-2.978	15.520
40	15.641	-1.927	13.714
100	15.004	-0.539	14.465
200	15.000	-0.065	14.935
$t \to \infty$	15.0	0.0	15.0

TABLE 1

The temperature at $x = 5$ is initially 10 degrees above steady-state; it falls more than a degree below that value before rising toward equilibrium. As the numbers illustrate, this behavior results from the fact that the temperature redistribution part, u_1, converges more rapidly than the heat change part, u_2.

Of course this isn't the end. The temperature distribution at intermediate times will be the sum of a curve from FIGURE 1 with a curve from FIGURE 2, and the resulting shape does not seem obvious. Further exploration is in order. A computer or calculator program to find temperatures at arbitrary times and places is not difficult to write. The only problem is accurately approximating the series; the comparison with a geometric series used above provides a crude but adequate error control. Such a program is also useful in exploring the effects of parameter changes.

References

[1] W. E. Boyce and R. C. DiPrima, Elementary Differential Equations & Boundary Value Problems, 3rd ed., J. Wiley, New York, 1977, pp. 483–486.

[2] F. B. Hildebrand, Advanced Calculus for Applications, 2nd ed., Prentice-Hall, New Jersey, 1976, pp. 461–463.

Characteristic Polynomials of Magic Squares

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Square matrices which are also magic squares have many fascinating properties and provide interesting problems for the linear algebra student. Several articles which discuss properties of matrix algebra and subspaces of magic squares are listed in our references; here we make some observations about characteristic polynomials of magic squares. The main result is a consequence of a beautiful but little-known theorem of Frobenius.

A **magic square** of order n is an $n \times n$ matrix in which the entries in each row, column and diagonal sum to the same number, called the **magic sum** of the matrix. An $n \times n$ matrix whose row and column sums are the same is called a **semi-magic square**, and the common sum the **semi-magic sum**. In the theory of characteristic polynomials it is convenient to work with matrices whose entries belong to an algebraically closed field, and hence we consider the vector space $\mathbf{A}(n)$ of semi-magic squares of order n with complex entries. The set $\mathbf{M}(n)$ of magic squares of order n is a subspace of $\mathbf{A}(n)$. Since $\mathbb{C}$ is algebraically closed, the characteristic polynomial $p_A(z)$ of an $n \times n$ matrix A is a product of linear factors

$$p_A(z) = \det(zI - A) = \prod_{i=1}^{n} (z - m_i), \tag{1}$$

where the complex numbers $m_1, \dots, m_n$ are the characteristic roots of A.

Suppose now that A is a semi-magic square of order n with semi-magic sum m. Since $Ae = me$, where $e = (1, \dots, 1)^T$, we see that m is an eigenvalue (and hence also a characteristic root) of A. In order to easily state our results, we let $m_1 = m$ in (1), and call $m_2, \dots, m_n$ in (1) the **complementary characteristic roots** of A. If we denote by E the magic square of order n all of whose entries are 1, then n is the magic sum of E. It is easy to see that 0 is a characteristic root of E of multiplicity $n-1$ (since rank $E = 1$, the dimension of the null space of E is $n-1$), hence the characteristic root n of E has multiplicity 1. Using our terminology, all of the complementary characteristic roots of E are 0. Note that if p is any complex number, then $A + pE$ is also semi-magic, with semi-magic sum $m + np$.

THEOREM 1. *If $A \in \mathbf{A}(n)$ and $p \in \mathbb{C}$, then A and $A + pE$ have the same complementary characteristic roots.*

The theorem is proved using the following result due to Frobenius (see [**3**], p. 22).

LEMMA (*Frobenius*). *Let A and B be $n \times n$ matrices for which $AB = BA$, let $a_1, \dots, a_n$ be the characteristic roots of A, and let $f(x, y)$ be a rational function. The characteristic roots of B can be ordered $b_1, \dots, b_n$ so that the characteristic roots of $f(A, B)$ are $f(a_1, b_1), \dots, f(a_n, b_n)$.*

Proof of Theorem 1: Let $A \in \mathbf{A}(n)$ have magic sum m, and $p \in \mathbb{C}$. We can apply the Lemma with $B = E$ and $f(x, y) = x + py$, since $AE = mE = EA$. Suppose that the characteristic roots of A are $m, m_2, \dots, m_n$. Since the semi-magic sum of $A + pE$ is $m + np$, we may order the characteristic roots of $A + pE$ as $m + np, k_2, \dots, k_n$. Since the characteristic roots of E are $n, 0, \dots, 0$, they can be ordered as $b_1, \dots, b_n$ such that

$$m + np = m + pb_1$$
$$k_i = m_i + pb_i \quad \text{for } 2 \leqslant i \leqslant n.$$

From the first equation we see that $b_1 = n$. Hence the other b_i's are zero and we see that $k_i = m_i$ for $2 \leqslant i \leqslant n$ as desired.

It is well known that the determinant of a matrix is the product of its characteristic roots and thus the theorem implies

$$\frac{\det(A+pE)}{m+np}=\frac{\det A}{m}. \tag{2}$$

The reader might wish to investigate similar relationships between the sum of the principal i-rowed minors of A and of $A+pE$ (Theorem 14.3 in [**3**], p. 19 is useful).

If A is as above but is a magic square (the diagonal entries also sum to m), then we have an additional constraint. Since the trace of a matrix is the sum of its characteristic roots

$$m=\operatorname{tr}(A)=m+m_2+\cdots+m_n, \tag{3}$$

we have the following result.

THEOREM 2. *The sum of the complementary characteristic roots of $A\in\mathbf{M}(n)$ is zero.*

Equation (1) shows that the coefficient of z^{n-1} in the characteristic polynomial of an $n\times n$ matrix is the sum of its characteristic roots. Theorem 2 implies that this coefficient is the magic sum of A. Thus the characteristic polynomial of $A\in\mathbf{M}(n)$ with magic sum m can be written in the form

$$(z-m)\left(z^{n-1}+a_{n-3}z^{n-3}+\cdots+a_0\right). \tag{4}$$

Our results can be applied to low-order matrices to provide interesting exercises. We illustrate for $n=3$. In [**6**] it is proved that $\dim\mathbf{M}(3)=3$, and it is a nice exercise to show that the matrices E,F,G, where

$$F=\begin{bmatrix}1&-1&0\\-1&0&1\\0&1&-1\end{bmatrix}\text{ and }G=\begin{bmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{bmatrix},$$

is a basis for $\mathbf{M}(3)$. If $A\in\mathbf{M}(3)$, then A can be written uniquely in the form

$$A=aE+bF+cG,\qquad a,b,c\in\mathbb{C}$$

and one can easily see that the magic sum of A is $m=3a$ and

$$\det A=9a(c^2-b^2). \tag{5}$$

If A' denotes the matrix of cofactors of A, a direct calculation shows that

$$A'=(c^2-b^2)E-3abF+3acG,$$

so $A'\in\mathbf{M}(3)$. But then $m'=3(c^2-b^2)$ is the magic sum of A' and (5) shows that $mm'=\det A$. If A is invertible, then $A^{-1}=(1/\det A)A'$, so $A^{-1}\in\mathbf{M}(3)$ and the magic sum of A^{-1} is $1/m$. This gives an alternate proof of results of Rose [**4**] and Lancaster [**1**], which state respectively that if A is an invertible magic square of order three, then A^{-1} is also a magic square of order three and the magic sum of A^{-1} is the reciprocal of the magic sum of A.

As a consequence of (2) and (4) we have: if $A\in\mathbf{M}(3)$, then, in the preceding notation, the characteristic polynomial of A is $(z-m)(z^2-m')$.

References

[1] R. J. Lancaster, Magic square matrices, Math. Teacher, 72 (1979) 30–32.
[2] N. J. Lord, The dimension of the space of magic squares, Math. Gazette, 66 (1982) 199–203.
[3] C. C. MacDuffee, The Theory of Matrices, Springer-Verlag, Berlin, 1933.
[4] D. M. Rose, Magic squares and matrices, Math. Gazette, 57 (1973) 36–39.
[5] B. D. Strand, Vector subspaces of magic squares, Pi Mu Epsilon Journal, 7 (1982) 430–438 and J. Undergrad. Math., 14 (1982) 5–10.
[6] J. E. Ward, III, Vector spaces of magic squares, this MAGAZINE, 53 (1980) 108–111.
[7] L. M. Weiner, The algebra of semi-magic squares, Amer. Math. Monthly, 62 (1955) 237–239.

Integral Solutions to the Equation $x^2 + y^2 + z^2 = u^2$: A Geometrical Approach

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In his *History of the Theory of Numbers*, Dickson [4] cites six identities, each of which gives infinitely many integral solutions of the equation

$$x^2 + y^2 + z^2 = u^2. \qquad (*)$$

These are:

(1) $(2pr)^2 + (2qr)^2 + (p^2 + q^2 - r^2)^2 = (p^2 + q^2 + r^2)^2$,
by V. A. Lebesgue [7];
(2) $[p(p+q)]^2 + [q(p+q)]^2 + (pq)^2 = (p^2 + pq + q^2)^2$,
by U. Dainelli [3];
(3) $[2qr(m^2 - n^2)]^2 + [(m^2 - n^2)(q^2 - r^2)]^2 + [2mn(q^2 + r^2)]^2 = [(m^2 + n^2)(q^2 + r^2)]^2$,
by C. Gill [6];
(4) $(4mp)^2 + [(m^2 - 1)(p^2 + 1)]^2 + [2m(p^2 - 1)]^2 = [(m^2 + 1)(p^2 + 1)]^2$,
by J. A. Euler [5];
(5) $(4m^2n^2)^2 + (m^4 - n^4)^2 + [2mn(m^2 - n^2)]^2 = [(m^2 + n^2)^2]^2$,
by the Japanese Matsunango [8]; and
(6) $q^2 + (q+1)^2 + [q(q+1)]^2 = (q^2 + q + 1)^2$,
by P. Cossali [2].

What we would like to do here is to show that these identities are special cases of a more general identity, by using analytical geometry of three dimensions.

If we divide equation $(*)$ by u^2, we get $(x/u)^2 + (y/u)^2 + (z/u)^2 = 1$. Now if x, y, z, and u are integers, then x/u, y/u, z/u are rational numbers, and our problem reduces to that of finding triples of rational solutions (x', y', z') to the equation

$$x'^2 + y'^2 + z'^2 = 1. \qquad (7)$$

The graph of this equation in E^3 is a sphere, and for m, $n \in Z$, the point

$$\left(\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2}, 0\right) \qquad (8)$$

lies on the sphere and represents a solution to equation (7). Of course $((m^2 - n^2)/(m^2 + n^2),$ $2mn/(m^2 + n^2))$ is the well-known solution to the equation $x'^2 + y'^2 = 1$. There are many derivations of this solution, one of them by H. Wright [9], in which he used plane coordinate geometry.

Now consider the line in E^3 through the point in (8), whose direction is given by the vector (r, p, q). The parametric equations for this line are:

$$x' = \frac{m^2 - n^2}{m^2 + n^2} + rt, \qquad y' = \frac{2mn}{m^2 + n^2} + pt, \qquad z' = qt,$$

where $r, p, q \in Z$, and t is a parameter. Substituting for x', y' and z' in equation (7) gives

$$\left(\frac{m^2 - n^2}{m^2 + n^2} + rt\right)^2 + \left(\frac{2mn}{m^2 + n^2} + pt\right)^2 + (qt)^2 = 1,$$

and solving for the parameter t, we find that $t = 0$ or

$$t=\frac{-2r(m^2-n^2)-4mnp}{(m^2+n^2)(p^2+q^2+r^2)}. \tag{9}$$

The value $t=0$ corresponds to the point in (8), and the second value of t, given by (9), corresponds to the other point at which the straight line intersects the surface of the sphere. Using the value of the parameter t in (9), and denoting by u its denominator, we get a new solution of equation $(*)$:

$$\begin{aligned} x &= (m^2-n^2)(p^2+q^2-r^2)-4mnpr \\ y &= 2mn(r^2-p^2+q^2)-2rp(m^2-n^2) \\ z &= -2qr(m^2-n^2)-4mnpq \\ u &= (m^2+n^2)(p^2+q^2+r^2). \end{aligned} \tag{10}$$

And now the following substitutions in (10) yield the identities cited in our opening paragraph:

(1) $m=1, \quad n=0.$
(2) $m=1, \quad n=0, \quad r=p+q.$
(3) $p=0.$
(4) $r=0, \quad n=q=1.$
(5) $r=0, \quad p=m, \quad q=n.$
(6) $m=p=1, \quad n=0, \quad r=1+q.$

Although the identity derived here provides more solutions to equation $(*)$ than any of the six given identities, it still does not give *all* the solutions of $x^2+y^2+z^2=u^2$. For example, $u=27$ is representable as $(m^2+n^2)(p^2+q^2+r^2)$ in two ways; either $m=1$, $n=0$, $p=q=r=3$ or $m=3$, $n=0$, $p=q=r=1$. The solution (10) to equation $(*)$ in either case is given by $18^2+18^2+9^2=27^2$. However there are other solutions to $(*)$ for $u=27$:

$$\begin{aligned} 23^2+14^2+2^2&=27^2\\ 26^2+7^2+2^2&=27^2\\ 22^2+7^2+14^2&=27^2. \end{aligned}$$

These three solutions can be obtained from the identity

$$(p^2+q^2-r^2-s^2)^2+[2(pr+qs)]^2+[2(ps-qr)]^2=(p^2+q^2+r^2+s^2)^2,$$

which provides the complete solution of $x^2+y^2+z^2=u^2$ as noted by E. Catalan [**1**].

Applying the same technique and making use of these identities (or special cases of them), one can derive identities that generate integral solutions to the equation $x^2+y^2+z^2+w^2=u^2$ or even to the more general equation $x_1^2+x_2^2+x_3^2+\cdots+x_n^2=u^2$.

References

[1] E. Catalan, Bull. Acad. Roy. Belgique, (3), 9 (1885) 531.
[2] P. Cossali, Origine, Transporto in Italia, Algebra, 1 (1797) 97.
[3] U. Dainelli, Giornale di Mat., 15 (1877) 378.
[4] L. E. Dickson, History of the Theory of Numbers, Volume 2, Chelsea Publishing Co., N.Y., 1966, pp. 260, 261, 262, 266, 269.
[5] J. A. Euler, Acta Acad. Petrop., 3 (1779) 40.
[6] C. Gill, The Gentleman's Math. Companion, London, 5, No. 29 (1826) 364.
[7] V. A. Lebesgue, Nouv. Ann. Math. (2), 13 (1874) 64.
[8] Y. Mikami, Abh. Geschichte Math. Wiss., 30 (1912) 233.
[9] H. N. Wright, First Course in Theory of Numbers, Dover Publications, New York, 1939.

Archimedes Revisited

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Archimedes proved the following geometric theorem (see Figure 1):

Let an arc p of a parabola with chord AB be given, and let C be the point on p where the tangent line is parallel to AB. If T is the area of triangle ABC and S is the area of the segment bounded by the arc p and the chord AB, then $T = \frac{3}{4}S$.

A discussion of Archimedes' proof can be found in [**1**]. A natural generalization of this classic result was given by M. Golomb and H. Haruki [**2**]:

Theorem. *If, in Archimedes' Theorem, p is the arc of any conic section, then*

$T/S < 3/4$ *if p is on an ellipse,*

$T/S = 3/4$ *if p is on a parabola,*

$T/S > 3/4$ *if p is on a branch of a hyperbola.*

The straightforward, but lengthy proof in [**2**] uses analytic geometry and calculus; an ingenious alternate proof by Haruki [**3**] uses conformal mapping. However, this could be considered as a complicated and unnatural method to prove the simple statement.

Affine transformations are linear and transform the line at infinity into itself. They leave invariant the three classes of conics and the set of nondegenerate triangles. In addition, affine transformations leave invariant the ratio of the areas of any two regions. Thus the theorem under consideration can be recognized as one of affine geometry, and it suffices to prove the theorem for one special case.

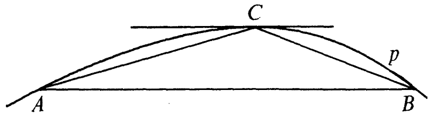

Figure 1

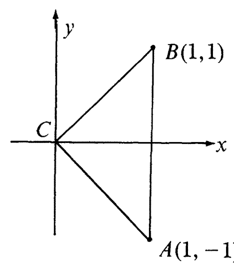

Figure 2

We choose triangle ABC to be the isosceles right triangle with vertices $A(1,-1)$, $B(1,1)$, $C(0,0)$ (see Figure 2). There is a pencil of conics passing through A and B and tangent at C to the y-axis. The parabola $c_1 \equiv y^2 - x = 0$ and the degenerate conic $c_2 \equiv x(x-1) = 0$ belong to the pencil. Hence its equation is $c_1 - kc_2 = 0$, or

$$y^2 = kx^2 + (1-k)x, \tag{1}$$

k being the parameter. The arc CB is given by $y \geqslant 0$, $0 \leqslant x \leqslant 1$, CA by $y \leqslant 0$, $0 \leqslant x \leqslant 1$. The configuration is symmetric with respect to the x-axis. The points at infinity of (1) are real and different for $k > 0$, coincide for $k = 0$, and are imaginary for $k < 0$. Hence the graph of equation (1) is an ellipse if $k < 0$ and a parabola if $k = 0$. For $k > 0$ it is a hyperbola, but we must restrict ourselves to $0 < k < 1$ (because for $k = 1$ the conic is degenerate and for $k > 1$ the three points are not on the same branch). For a parabola ($k = 0$ in (1)) we have

$$S = 2\int_0^1 x^{1/2}\,dx = 2 \cdot \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{4}{3} = \frac{4}{3}T, \tag{2}$$

which is Archimedes' result.

If y_k denotes the arc CB of equation (1) and y_0 the parabolic arc, we have for $0 < x < 1$

$$y_k^2 - y_0^2 = kx^2 + (1-k)x - x = -kx(1-x), \tag{3}$$

which is positive for an ellipse ($k < 0$) and negative for a hyperbola ($0 < k < 1$). Hence when the arc CB is on an ellipse, it is above the parabolic arc y_0 and when CB is on a hyperbola it lies below y_0. This concludes the proof of the theorem.

References

[1] E. J. Dÿksterhuis, Archimedes, Acta historica scientiarum naturalium et medicinalium, 12 (1956), Copenhagen, 336–345.

[2] M. Golomb and H. Haruki, An inequality for elliptic and hyperbolic segments, this MAGAZINE, 46 (1973) 152–155.

[3] H. Haruki, An application of conformal mapping to an inequality for elliptic and hyperbolic segments, Mathematicae Notae, 27 (1979–80) 15–22.

Solution of a Conjecture Concerning Air Resistance

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In a note in this MAGAZINE [**1**], Lekner discusses the motion of a particle projected vertically in a uniform gravitational field and subject to an air resistance (drag) depending only on the particle's speed and obtains the equation

$$\tau = \frac{1}{2u_i}\left[\int_0^{u_i}\frac{du}{1+\phi(u)} + \int_0^{u_f}\frac{du}{1-\phi(u)}\right] \tag{1}$$

where

$$\int_0^{u_f}\frac{u\,du}{1-\phi(u)} = \int_0^{u_i}\frac{u\,du}{1+\phi(u)}. \tag{2}$$

If g denotes acceleration due to gravity and v_t is the terminal speed of the particle (the one speed at which the drag and gravitational forces balance), then $v_t u_i$ is the projection (initial) speed, $v_t u_f$ is the return (final) speed (it is assumed $u_f < 1$), $v_t u$ is the general speed, $g\phi(u)$ is the deceleration due to air drag (so $\phi(0) = 0$ and $\phi(1) = 1$) and τ is the ratio of the return time (the total time elapsed from the instant of projection until the particle returns to the projection point) to the return time under zero drag conditions. Equation (2) represents equality of the ascent and descent distances.

For the special case $\phi(u) = u^p$, p a positive constant, Lekner conjectures the following properties of τ (C2(iii) in modified form).

C1. $\tau < 1$ for all initial speeds if $p \geqslant 1$.

C2. If $p < 1$, then

(i) $\tau < 1$ for sufficiently small initial speeds,

(ii) $\tau > 1$ for sufficiently large initial speeds, and

(iii) $\tau = 1$ for some u_i and for the least such u_i, $u_i \to \infty$ as $p \to 1$.

which is Archimedes' result.

If y_k denotes the arc CB of equation (1) and y_0 the parabolic arc, we have for $0 < x < 1$

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(ii) $\tau > 1$ for sufficiently large initial speeds, and
(iii) $\tau = 1$ for some u_i and for the least such u_i, $u_i \to \infty$ as $p \to 1$.

We shall establish some general results which imply the truth of C1 and C2. Klamkin [**2**] has also established generalizations of C1 and C2(i) and (ii) under physically reasonable restrictions on ϕ. In the apparent absence of any simple physical explanation of C1 and C2, we use a purely mathematical approach in which a number of standard real analysis results are employed without comment. For notational convenience we write $x = u_i$ and $y = u_f$.

The following hypotheses are adopted for ϕ; they reflect the physical basis of the conjecture and are satisfied when $\phi(u) = u^p$, $p > 0$. *The real-valued function* $\phi = \phi(u)$ *is assumed to be continuous and (strictly) increasing on* $[0, \infty)$ *with* $\phi(0) = 0$, $\phi(1) = 1$, *and left-differentiable at* $u = 1$. The differentiability condition ensures the divergence of $\int_0^1 u/(1-\phi(u))\,du$, this divergence corresponding to the physical assumption that the terminal speed is never reached during descent. Under these conditions ϕ is positive on $(0, \infty)$ and has a continuously increasing inverse ϕ^{-1} defined on at least $[0,1]$ with $\phi^{-1}(0) = 0$ and $\phi^{-1}(1) = 1$.

THEOREM 1. *Under the stated hypotheses for* ϕ, *there exists a unique function* $y = y(x)$, $x \geqslant 0$, *with* $0 \leqslant y < 1$, *satisfying the integral equation*

$$\int_0^y \frac{u\,du}{1-\phi(u)} = \int_0^x \frac{u\,du}{1+\phi(u)} \tag{3}$$

and the function so defined has the properties

(*i*) $y = 0$ *when* $x = 0$ *and* $0 < y < x$ *otherwise*,
(*ii*) $y^2/(1-\phi(y)) > x^2/(1+\phi(x))$ *when* $x > 0$, *and*
(*iii*) $y(x)$ *is differentiable for* $x > 0$ *with* $y'(x) = (x/y)(1-\phi(y))/(1+\phi(x))$.

Proof. We can define nonnegative, continuously increasing functions H_- and H_+ by

$$H_-(y) = \int_0^y \frac{u\,du}{1-\phi(u)}, \qquad 0 \leqslant y < 1, \tag{4}$$

and

$$H_+(x) = \int_0^x \frac{u\,du}{1+\phi(u)}, \qquad x \geqslant 0, \tag{5}$$

noting that H_-^{-1} exists with range $[0,1)$. Now

$$0 < \frac{1-\phi(u)}{1-u} = \frac{\phi(1)-\phi(u)}{1-u} \leqslant \frac{1}{k}$$

for some constant $k > 0$ and $0 \leqslant u < 1$ since the hypotheses for ϕ imply $(\phi(1)-\phi(u))/(1-u)$ is positive and continuous on $[0,1)$ and has a limit as $u \to 1^-$. Therefore

$$H_-(y) = \int_0^y \frac{1-u}{1-\phi(u)}\left[\frac{u}{1-u}\right] du \geqslant k\int_0^y \frac{u\,du}{1-u^2} = \frac{k}{2}\log\frac{1}{1-y^2}$$

so $H_-(y) \to \infty$ as $y \to 1$; the intermediate value property of continuous functions implies that the range of H_- (and thus the domain of H_-^{-1}) is $[0,\infty)$. Hence the composition $H_-^{-1} \circ H_+$ exists and setting $y = H_-^{-1} \circ H_+(x)$, $x \geqslant 0$, we see that $0 \leqslant y < 1$ and $H_-(y) = H_+(x)$, which is just (3). Conversely, if $H_-(y) = H_+(x)$ then $y = H_-^{-1} \circ H_+(x)$; this establishes existence and uniqueness.

From (3), $y = 0$ if $x = 0$ and if $x > 0$, then $0 < \int_0^y u\,du < \int_0^x u\,du$ which gives (i) and similarly $1/(1-\phi(y))\int_0^y u\,du > 1/(1+\phi(x))\int_0^x u\,du$ which gives (ii).

From (4) and (5), it is clear H_- and H_+ each has a positive derivative except at zero so $y = H_-^{-1} \circ H_+(x)$ is differentiable for $x > 0$, and since $H_-(y) = H_+(x)$ we conclude that $H_-'(y)y'(x) = H_+'(x)$ for $x > 0$; this gives (iii) and the proof is complete.

With $y = y(x)$ defined by (3), define

$$\tau = \tau(x) = \frac{1}{2x}\left[\int_0^x \frac{du}{1+\phi(u)} + \int_0^y \frac{du}{1-\phi(u)}\right], \qquad x > 0. \tag{6}$$

Then τ is positive, differentiable and bounded as $x \to 0$ since

$$\tau < \frac{1}{2x}\left[\int_0^x du + \frac{1}{1-\phi(y)}\int_0^y du\right] = \frac{1}{2}\left[1 + \frac{y/x}{1-\phi(y)}\right] < \frac{1}{2}\left[1 + \frac{1}{1-\frac{1}{2}}\right] = \frac{3}{2}$$

whenever $0 < \phi(x) < \frac{1}{2}$.

THEOREM 2. *With* τ *defined as in* (6),

(*i*) $\tau < 1$ *whenever* $0 < \phi(x) \leqslant 2^{-1/2}$,
(*ii*) $\tau < 1$ *for all* $x > 0$ *if* $\phi(u)/u$ *is nondecreasing on* $(0, \infty)$, *and*
(*iii*) $\tau > 1$ *for sufficiently large* x *if* $\phi(u)/u \leqslant k < \frac{1}{2}$ *for some* (*positive*) *constant* k *and sufficiently large* u.

Proof. We suppose $x > 0$ throughout. By (6) and Theorem 1(iii),

$$\frac{d}{dx}(2x\tau) = \frac{1}{1+\phi(x)} + \frac{y'(x)}{1-\phi(y)} = \frac{1+x/y}{1+\phi(x)}$$

and, therefore,

$$\frac{d}{dx}(2x\tau) < 2 \quad \text{if } y > w, \quad \text{where } w = \frac{x}{1+2\phi(x)}. \tag{7}$$

If $y \leqslant w$, then $x^2(1-\phi(w)) \leqslant x^2(1-\phi(y))$ and $y^2(1+\phi(x)) \leqslant w^2(1+\phi(x))$. But $x^2(1-\phi(y)) < y^2(1+\phi(x))$ by Theorem 1(ii), so $x^2(1-\phi(w)) < w^2(1+\phi(x))$ if $y \leqslant w$, and hence $y > w$ if $x^2(1-\phi(w)) \geqslant w^2(1+\phi(x))$. Rearranging this last inequality and applying (7) we conclude that

$$\frac{d}{dx}(2x\tau) < 2 \quad \text{if } (*)\ \phi(w)(1+2\phi(x))^2 \leqslant \phi(x)(3+4\phi(x)). \tag{8}$$

Now $w < x$, so $(*)$ holds when $\phi(x)(1+2\phi(x))^2 \leqslant \phi(x)(3+4\phi(x))$, which reduces to $\phi(x) \leqslant 2^{-1/2}$. Thus by (8), $(d/dx)(2x\tau) < 2$, i.e., $(d/dx)(2x(\tau-1)) < 0$, if $\phi(x) \leqslant 2^{-1/2}$, and hence $2x(\tau-1)$ is decreasing on $(0, \phi^{-1}(2^{-1/2})]$. But τ is bounded as $x \to 0$, so $2x(\tau-1) \to 0$ as $x \to 0$ and therefore $2x(\tau-1) < 0$, i.e., $\tau < 1$, if $0 < \phi(x) \leqslant 2^{-1/2}$, and (i) is proved.

For (ii) we note $(*)$ holds if $\phi(w)(1+2\phi(x))^2 \leqslant \phi(x)(2+4\phi(x))$, i.e., if $\phi(w) \leqslant 2w\phi(x)/x$ so (8) gives

$$\frac{d}{dx}(2x\tau) < 2 \quad \text{if } \frac{\phi(w)}{w} \leqslant \frac{2\phi(x)}{x}, \tag{9}$$

and the truth of (ii) follows from (9) since $w < x$.

By the conditions of (iii) we can choose $x_0 > 1$ such that $\phi(u)/u \leqslant k$ and $1/u \leqslant \frac{1}{2}(\frac{1}{2} - k)$ whenever $u \geqslant x_0$, and these inequalities imply

$$\frac{u}{1+\phi(u)} \geqslant \frac{2}{\frac{1}{2}+k} \quad \text{for all } u \geqslant x_0. \tag{10}$$

Hence for $x \geqslant x_0/(\frac{1}{2} - k)$, i.e., whenever $1 - (x_0/x) \geqslant \frac{1}{2} + k$, we have, by (6), (3) and (10),

$$\begin{aligned}\tau &> \frac{1}{2x}\int_0^y \frac{du}{1-\phi(u)} > \frac{1}{2x}\int_0^y \frac{u\,du}{1-\phi(u)} = \frac{1}{2x}\int_0^x \frac{u\,du}{1+\phi(u)} \\ &> \frac{1}{2x}\int_{x_0}^x \left[\frac{2}{\frac{1}{2}+k}\right] du = \frac{1-x_0/x}{\frac{1}{2}+k} \geqslant 1,\end{aligned}$$

and the proof is complete.

Remark. Using $\phi(0) = 0$ and the definition of convexity, it is easy to show $\phi(u)/u$ is non-decreasing on $(0, \infty)$ if ϕ is convex on $[0, \infty)$. Hence by Theorem 2(ii), $\tau < 1$ for all $x > 0$ if ϕ is convex on $[0, \infty)$; this result is obtained by Klamkin [**2**], who also obtains Theorem 2(iii) (in both cases by arguments using derivatives) and a weaker version of Theorem 2(i).

With the above results in hand, the truth of C1 and C2 follows easily: C2(i) is immediate by Theorem 2(i) and since, on $(0,\infty)$, $u^p/u = u^{p-1}$ is non-decreasing when $p \geqslant 1$ and decreasing to zero when $p<1$, C1 and C2(ii) follow by Theorem 2(ii) and (iii), respectively. For C2(iii) we set $\phi(u)=u^p$, $0<p<1$, and apply (9): some routine algebra shows that $\phi(w)/w \leqslant 2\phi(x)/x$ if $(1+2x^p)^{1-p} \leqslant 2$, so from (9)

$$\tau<1 \quad \text{if } 0<x\leqslant x_p, \quad \text{where } x_p = \left[2^{p/(1-p)} - \tfrac{1}{2}\right]^{1/p} \tag{11}$$

Thus by (11), C2(ii) and the continuity of τ, there is a least (and a greatest) $x > x_p$ for which $\tau = 1$ and since $x_p \to \infty$ as $p \to 1$, the truth of C2 follows.

The above analysis of τ, initially motivated by a physical situation, suggests some purely mathematical questions. In C2(iii), is there *one* u_i for which $\tau = 1$? What is the general behaviour of the graph of τ in the cases $\phi(u)=u^p$? Can we draw any general conclusions about the behaviour of τ when ϕ is concave? Perhaps some computer analysis would be useful in suggesting possible answers. Asymptotic properties of τ seem easier to establish; we mention one which complements Theorem 2(iii): $\tau<1$ for sufficiently large x if $\phi(u)/u \geqslant k > \frac{1}{2}$ for some constant k and sufficiently large u, with this result extending to $k=\frac{1}{2}$ if $\int_0^1 (1-u)/(1-\phi(u))\,du$ converges (e.g., if $\phi(u)$ has a non-zero left derivative at $u=1$). We conclude by noting that property (i) in Theorem 1 means that the projection speed always exceeds the return speed (also shown in [**1**] by conservation of total energy). This suggests the following analogous results which, at least under the hypotheses for ϕ we have used, students may wish to establish both from a physical and purely mathematical viewpoint: the descent time and ascent energy dissipation (the work done against air resistance) always exceed the ascent time and descent energy dissipation, respectively.

References

[1] J. Lekner, What goes up must come down; will air resistance make it return sooner, or later?, this MAGAZINE, 55 (1982) 26–28.

[2] M. S. Klamkin, Vertical Motion with Air Resistance, Mathematical Modelling in Science and Technology, The Fourth International Conference, Pergamon Press, New York, 1984, pp. 993–995.

A Differentiation Test for Absolute Convergence

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In this note, we describe a new test which provides a necessary and sufficient condition for absolute convergence of infinite series. The test is based solely on differentiation and is very easy to apply. It also provides a pictorial illustration for absolute convergence and divergence.

The discovery was made a few years ago when I was asked to give a lecture on infinite series to my classmates. The subject was new to us and some basic ideas were not quite appreciated at that early stage. I tried to give an informal or pictorial illustration for any concept that sounded abstract. When I mentioned that an infinite series would converge absolutely if its *far away* terms became *small enough*, I had to explain to the students (and to myself as it turned out) *how* far and *how* small.

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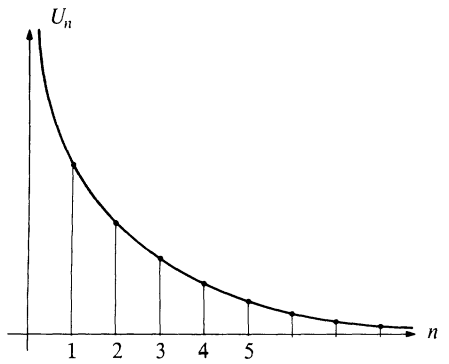

FIGURE 1

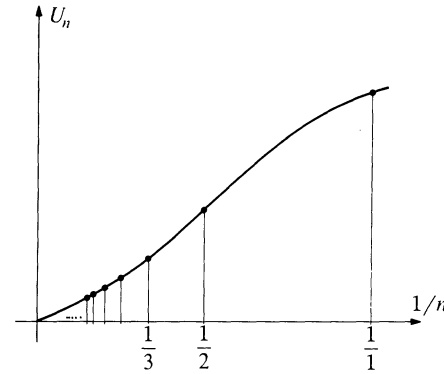

FIGURE 2

Plotting the series term by term with the summation index running along the x-axis was not very successful (FIGURE 1). I never seemed to get far enough and the terms looked quite small even with the harmonic series which, I had recently learned, was divergent. I had to come up with a picture that would show what a small term is and how it remains small if one multiplies the whole series by 10^{100} and why no matter how far away a point is, it may still not be far enough.

Remembering the duality of zero and infinity, I thought it might be nice to plot the series term by term with the *inverse* of the summation index running along the x-axis (FIGURE 2) so that we see the whole thing crowded near zero. First of all, the curve of any convergent series had to "hit" the origin since the terms go to zero as $n \to \infty$. I felt that the shape of the curve near the origin should also be related to convergence.

When I plotted divergent series like $\Sigma 1/n$ and $\Sigma 1/\sqrt{n}$, I ended up with positive or infinite slopes at the origin, but the convergent series $\Sigma 1/n^2$ had slope zero at the origin (FIGURE 3). The correspondence between the zero slope and the idea of "small terms" was appealing since a zero slope multiplied by 10^{100} is still a zero slope. The fact that the slope of a curve is a limit concept was in accordance with the "far away" idea. After the lecture, I rushed to check whether my particular illustration might generalize. After some scribbling, it gave rise to a valid criterion for absolute convergence which I call the **differentiation test**.

You probably have guessed the mechanism of the test by now. Roughly speaking, you take the infinite series in question, ΣU_n, and construct the function f defined by $f(1/n) = U_n$. First check that $f(0) = 0$. Now differentiate f and check the value of $f'(0)$: if this is also zero, the series is convergent.

Let us state this formally with the proper qualifying conditions:

DIFFERENTIATION TEST. *Let $\Sigma_{n=1}^{\infty} U_n$ be an infinite series with real terms. Let $f(x)$ be any real function such that $f(1/n) = U_n$ for all positive integers n and d^2f/dx^2 exists at $x = 0$. Then $\Sigma_{n=1}^{\infty} U_n$ converges absolutely if $f(0) = f'(0) = 0$ and diverges otherwise.*

Notice that there is a requirement that $f''(0)$ exists for the test to apply. When this requirement is satisfied, the test is guaranteed to determine whether the series is absolutely convergent or divergent. We will say more about relaxing the requirement of existence for $f''(0)$ later on, but first we present some examples to see how the test works.

We start with simple examples in which we know whether or not the series is convergent (after all, we haven't proved anything yet). Consider the two series

$$\sum_{n=1}^{\infty} \sin\frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} 1 - \cos\frac{1}{n}.$$

There are many ways to verify that the first series is divergent while the second is convergent. Applying the differentiation test, we examine the functions $\sin x$ and $1 - \cos x$, both of which have second derivatives at $x = 0$. The test now tells us that the first series is divergent

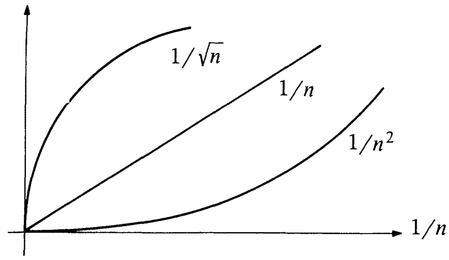

FIGURE 3

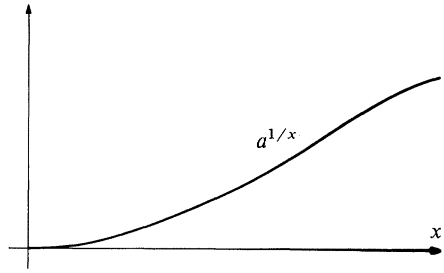

FIGURE 4

($f'(0) = 1 \neq 0$) and the second is absolutely convergent ($f(0) = f'(0) = 0$).

Another interesting example is the geometric series $\Sigma_{n=1}^{\infty} a^n$ where $0 < a < 1$. When we substitute x for $1/n$, we get the function $f(x) = a^{1/x} = e^{(\ln a)/x}$ (for $x > 0$ and zero otherwise). Since $\ln a < 0$, $f(x)$ goes to zero very quickly as $x \to 0$ (FIGURE 4). In fact, $f(x)$ has the derivatives of *all* orders at $x = 0$ equal to zero. To see this, differentiate $f(x)$ any number of times and you will always get a (finite) polynomial in $1/x$ multiplied by $f(x)$ itself. Since the exponential is "stronger" than any polynomial, all the derivatives will go to zero as $x \to 0$. This suggests that the geometric series with $0 < a < 1$ is *very* convergent, which is indeed the case.

Now we show some examples where you can determine convergence or divergence right away using the differentiation test while others will require effort to get the result. In fact, for the following examples, using other techniques to determine convergence is practically the same as writing the proof for the differentiation test in the general case.

Consider the infinite series

$$\sum_{n=1}^{\infty} \int_0^{1/n} g(t)\, dt$$

where $g(t)$ is any function that has a derivative at $t = 0$. You are required to determine the conditions on g for the series to converge absolutely. How long does it take you to conclude that it will converge absolutely if, and only if, $g(0) = 0$? To verify the result, try substituting simple functions like t^2 or e^{-t} for $g(t)$ and carry out the integration, then test the resulting series for convergence or divergence using standard methods.

Now consider:

$$\sum_{n=1}^{\infty} \sinh\left(\tanh\frac{1}{n} - \tan\frac{1}{n} + \sec\frac{1}{n^2} - \cosh\frac{1}{n}\right).$$

Since $\sinh(\tanh x - \tan x + \sec x^2 - \cosh x)$ is analytic and has zero value and zero derivative at $x = 0$, the series is absolutely convergent. You can try other compositions of simple functions like these and see that the differentiation test is equivalent to expanding the composite function $f(x)$ in a Taylor series about $x = 0$ and checking that the lowest power in the expansion is at least x^2.

If you would like to see other techniques for dealing with these examples, go through the following proof of the differentiation test which depends on such techniques.

Proof. Our proof of the differentiation test depends on L'Hospital's rule, the limit comparison test [**1**], and the integral test.

Since d^2f/dx^2 is assumed to exist at $x = 0$, we are guaranteed (among other things) that $f(x)$ is continuous at $x = 0$ and is differentiable in a neighborhood of $x = 0$ (we will need the latter to apply L'Hospital's rule). We thus have the following steps relating the conditions on $f(x)$ to the absolute convergence of $\Sigma_{n=1}^{\infty} U_n = \Sigma_{n=1}^{\infty} f(1/n)$:

(1) $f(0) = 0$ is necessary for any convergence, since $\lim_{n \to \infty} U_n = \lim_{x \to 0} f(x) = f(0)$ and if this is non-zero, the series must diverge.

(2) Suppose that $f(0)$ does equal zero, but $f'(0) = a \neq 0$. Then $\lim_{x \to 0} f(x)/x$

$= \lim_{x \to 0}(f(x) - f(0))/(x - 0) = a$. Consequently, $\lim_{n \to \infty}|U_n|/(1/n) = |a| \neq 0$. By the limit comparison test, $\Sigma_{n=1}^{\infty} U_n$ diverges absolutely since the harmonic series also does.

(3) We have determined that $f(0) = f'(0) = 0$ is necessary for convergence. We now assume that this condition holds and prove sufficiency. Take $0 < u < 1$ and consider the limit

$$\lim_{x \to 0^+} \frac{f(x)}{x^{1+u}} = \lim_{x \to 0^+} \frac{f'(x)}{(1+u)x^u} = \frac{1}{1+u} \lim_{x \to 0^+} \left(\frac{f'(x) - f'(0)}{x - 0} \right) x^{1-u} = \frac{f''(0)}{1+u} \lim_{x \to 0^+} x^{1-u} = 0,$$

where the first equality is an application of L'Hospital's rule. Therefore, $\lim_{n \to \infty}|U_n|/(1/n)^{1+u} = 0$ and again by the limit comparison test, $\Sigma_{n=1}^{\infty} U_n$ must converge absolutely since $\Sigma_{n=1}^{\infty} 1/n^{1+u}$ converges absolutely by the integral test.

Steps (1), (2), (3) complete the proof.

Perhaps you noticed in part (3) of the proof that the convergence did not depend critically on the existence of $f''(0)$. This is indeed the case and the existence of $f''(0)$ can be replaced by a weaker condition. We note that the condition cannot be completely removed since $\Sigma_{n=2}^{\infty} 1/n \ln n$, which is absolutely divergent by the integral test, has terms $f(1/n)$ where $f(x) = -x/\ln x$ (for $x > 0$ and zero otherwise); this function $f(x)$ has zero value and zero derivative at $x = 0$, but a non-existent second derivative.

The existence of $f''(0)$ in the differentiation test can be replaced, for example, by the existence of $\lim_{x \to 0^+} f'(x)/x^u$ or $d^2|x|^u f(x)/dx^2|_{x=0}$ for some $0 < u < 1$ (both conditions are implied by the existence of $f''(0)$ when $f(0) = f'(0) = 0$). Very minor modification of part (3) of the proof above is needed in these cases. Certain weaker conditions will also work; their discovery is left as a simple exercise.

It is also obvious that only the existence of $f'(0)$ is needed to conclude absolute divergence of a series using the test. Since divergence is seldom good news, I choose to leave the test in its simple symmetric form. Finally, one can apply the test with $f(1/n) = |U_n|$ instead of U_n. This covers complex series as well.

I should like to acknowledge Dr. Brent Smith and one of the referees for their assistance.

References

[1] T. M. Apostol, Mathematical Analysis, Addison-Wesley, 1974, pp. 183–193.
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Proof without Words: Algebraic areas

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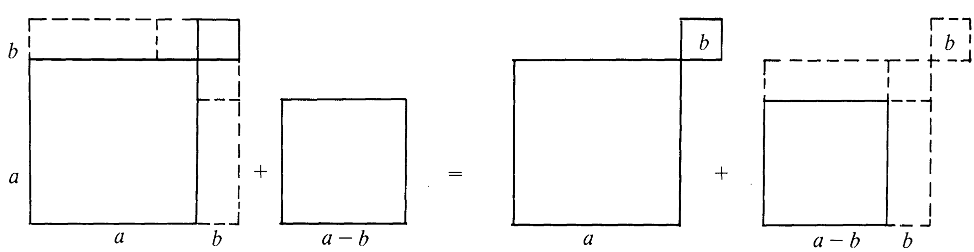

—Shirley Wakin
University of New Haven

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$$\lim_{x \to 0^+} \frac{f(x)}{x^{1+u}} = \lim_{x \to 0^+} \frac{f'(x)}{(1+u)x^u} = \frac{1}{1+u} \lim_{x \to 0^+} \left(\frac{f'(x) - f'(0)}{x - 0} \right) x^{1-u} = \frac{f''(0)}{1+u} \lim_{x \to 0^+} x^{1-u} = 0,$$

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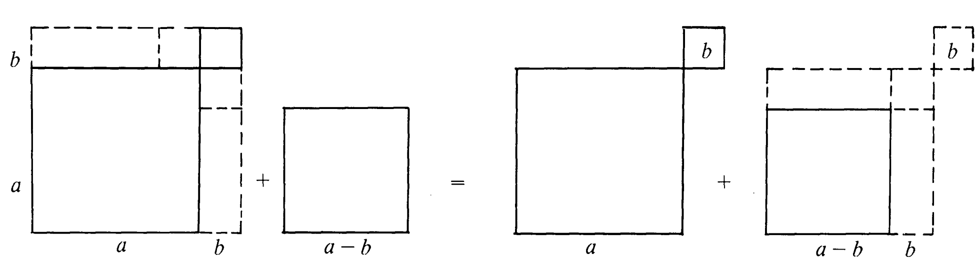

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Divisibility of Polynomial Expressions

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When I first saw Fermat's Last Theorem I decided to try and prove it (as I guess everyone does at least once). Not surprisingly, I wasn't successful, but in the course of considering polynomial expressions in two variables, I was led to the following theorem (in which the greatest common divisor of x and y is denoted (x, y)):

THEOREM. *Suppose a, b, and c_i, $i = 0, 1, \ldots, n$ are integers and $(a, b) = 1$. Then*

$$\left(a-b, c_0a^n + c_1a^{n-1}b + \cdots + c_{n-1}ab^{n-1} + c_nb^n\right) = (a-b, c_0 + c_1 + \cdots + c_n)$$

for any positive integer n.

The theorem can be stated in other more familiar forms, but the surprising disappearance of the a's and b's on the right-hand side makes it notable, I think, in this form. Also, it has as corollaries several well-known results, and some not so well known.

Proof. (By induction.) Consider first the case $n = 1$. Applying the fact that makes the Euclidean Algorithm work—that if $A = BQ + R$ then $(A, B) = (B, R)$—from

$$c_0a + c_1b = c_0(a-b) + b(c_0 + c_1)$$

we get

$$r = (a-b, c_0a + c_1b) = (a-b, b(c_0 + c_1)).$$

The only primes which can be factors of r are factors of $c_0 + c_1$: if $p|r$, then $p|(a-b)$ and so $p|b$ is impossible since $(a, b) = 1$. Thus

$$r = (a-b, c_0 + c_1).$$

Suppose now that the theorem is true for all positive integers $m \leqslant n$. Then

$$\begin{aligned}&\left(a-b, c_0a^{n+1} + c_1a^nb + \cdots + c_{n+1}b^{n+1}\right)\\ &\quad = \left(a-b, a\left[c_0a^n + c_1a^{n-1}b + \cdots + c_nb^n\right] + b\left[c_{n+1}b^n\right]\right)\\ &\quad = \left(a-b, c_0a^n + c_1a^{n-1}b + \cdots + c_{n+1}b^n\right)\end{aligned}$$

which follows because by the inductive hypothesis the theorem is true for $m = 1$. But we also know the theorem is true for $m = n$. Hence the above is equal to $(a-b, c_0 + c_1 + \cdots + c_n + c_{n+1})$, which proves the theorem for all positive integers n.

As one immediate corollary, we have that if

$$c_0 + c_1 + \cdots + c_n = 0,$$

then

$$(a-b)|\left(c_0a^n + c_1a^{n-1}b + \cdots + c_nb^n\right).$$

As another, if $c_i = \binom{n}{i}$, then $(a-b, (a+b)^n) = (a-b, 2^n)$.

If we take $a = 10$ and $b = 1$, then $c_0a^n + c_1a^{n-1}b + \cdots + c_nb^n$ is the base-10 representation of an integer N with digits $c_0, c_1, \ldots, c_n$ and the theorem says that 9 divides N if and only if 9 divides the sum of the digits of N, the old rule of Casting Out Nines. Take $a = 10$ and $b = -1$ to get the criterion for divisibility by 11. Similar criteria for divisibility by 101 or 99 come from choosing $a = 100$ and $b = 1$ or -1. By taking $b = 1$, analogous results can be found in any base.

For an application to Fermat's Last Theorem, suppose p is an odd prime and $x^p + y^p = z^p$, with $(x, y) = (x+y, p) = 1$. Then the theorem, with $a = x$ and $b = -y$, gives

$$(x+y, x^{p-1}-x^{p-2}y+\cdots+y^{p-1})=(x+y, p)=1.$$

Thus, since the product of the two factors on the left is x^p+y^p, a perfect pth power, both must be perfect pth powers, so $x+y=s^p$ for some integer s. The theorem can also be applied when $(x+y, p)=p$ to get a similar result, namely, $x+y=s^p p^k$. Thus, if Fermat's Theorem is false, $x+y$ must be very large.

A Note on Conditional Probabilities

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There seems to be little doubt that the most important concepts in probability theory are those related to conditioning. It is therefore important to have an unambiguous definition of the number $P(A|B)$, the probability that A occurs "given the event B". The definition that makes sense in the classical Kolmogorov-type model of probability (in which only individual probabilities are assumed), and the one most often used is this:

Definition. $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$ for all events with $P(B)\neq 0$, and is not defined if $P(B)=0$.

The question we wish to address in this note is: *why is the classical definition of $P(A|B)$ reasonable*? What is really behind this question is the desire to better understand the concept of conditioning.

We first must try to analyze the statement "given the event B" which is an integral part of the situation. Clearly, events cannot be given or taken away; thus we must mean something else. The general consensus as to what is meant when we say "find the probability of A given B" is "assume that B has occurred and find the probability that A will occur". We shall adhere to this point of view and the following assumption: whenever an event B occurs (for which $P(B)\neq 0$), the probabilities of all events will be affected. Thus the original probability measure P must be replaced by a new measure P_B; this is what we call the **process of conditioning**. The question of *how* P and P_B are related is answered by the classical formula in the definition above.

From this point of view it is obvious that $P_B(A)$ must be somehow related to $P(A\cap B)$ because "given B" the event A occurs if and only if $A\cap B$ occurs. We claim that the classical formula is too simple a relation to accept without some thought; on the other hand, we do not wish to offer justifications based on quasi-empirical arguments which involve frequencies of occurrence in successive trials. Instead, we intend to show how the classical definition follows from three simple qualitative hypotheses on the process of conditioning.

First, we need some background. We assume we are working within the Kolmogorov model of probability theory, which consists of a set X, a σ-algebra $\mathscr{E}$ of subsets of X representing the events we are studying, and a positive measure P on $\mathscr{E}$ which assigns to each event E its probability $P(E)$, and for which $P(X)=1$. The pair $(X,\mathscr{E})$ is called a **probability space**. When $\mathscr{E}$ is finite, $(X,\mathscr{E})$ is called a **finite probability space**. If $\mathscr{E}$ is finite, it is not hard to see that there exist a finite number of events $A_1, A_2, \ldots, A_N\in\mathscr{E}$ (called atoms of $\mathscr{E}$) such that every event except $\varnothing$

$$(x+y, x^{p-1}-x^{p-2}y+\cdots+y^{p-1})=(x+y,p)=1.$$

Thus, since the product of the two factors on the left is x^p+y^p, a perfect pth power, both must be perfect pth powers, so $x+y=s^p$ for some integer s. The theorem can also be applied when $(x+y,p)=p$ to get a similar result, namely, $x+y=s^p p^k$. Thus, if Fermat's Theorem is false, $x+y$ must be very large.

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is a union of certain of these A_i. Thus an event is an atom if the only event properly contained in it is the empty (i.e., the impossible) event. Clearly, any two distinct atoms are disjoint (mutually exclusive).

A measure which we call the counting (probability) measure on the finite space $(X,\mathscr{E})$ assigns to each $E\in\mathscr{E}$ the number k/N, where k is the number of atoms contained in E and N is the number of atoms in $\mathscr{E}$. This measure describes the situation where all atoms have the same probability of occurrence (sometimes referred to as the totally random situation).

One more fact will be of use to us. Suppose $(X,\mathscr{E})$ is a finite probability space with atoms $A_1, A_2,\dots, A_N$ and $(Y,\mathscr{F})$ is a second probability space (not necessarily finite) which has a set $E_1, E_2,\dots, E_N$ of pairwise disjoint events in $\mathscr{F}$ such that $\bigcup_{i=1}^N E_i = Y$. Then the map

$$\rho: \bigcup_{i\in F} A_i \to \bigcup_{i\in F} E_i$$

where F is any finite subset of $\{1,2,\dots,N\}$ is well defined and a homomorphism of $\mathscr{E}$ into $\mathscr{F}$ in the sense that ρ preserves unions, intersections and complements.

The hypotheses on conditioning and consequences

Given an arbitrary probability space $(X,\mathscr{E})$ and an arbitrary probability measure P on $\mathscr{E}$, we assume that to each event B with $P(B)\neq 0$ there corresponds a probability measure P_B on $\mathscr{E}$. We shall also write, as per the standard convention, $P(E|B)$ for the number $P_B(E)$, which is supposed to be the probability that event E occurs, given that B has occurred. Our aim is to compute P_B in terms of the original measure P.

We make the following hypotheses:

(i) For any $B\in\mathscr{E}$ with $P(B)\neq 0$ we have $P(B|B)=1$.

(ii) If P is a counting probability measure, then P_B assigns equal probabilities to the atoms contained in B.

(iii) $P(E|B)$ is a continuous function of the probabilities of the events in the algebra generated by the events B and E. Furthermore, $P(E|B)$ has the same definition regardless of the probability space or measure.

Before we discuss the consequences of these hypotheses, a few comments are in order.

Hypothesis (i) is the most innocent, simply stating the obvious. Hypothesis (ii), which refers to a very special situation, says that the occurrence of B produces the least possible disturbance in the distribution of the probabilities of atoms of B. Obviously, the classical formula implies this. We note that the first two hypotheses imply the classical formula for the case of counting probability measures. To see this, observe that (i) implies that $P_B(A)=0$ for every atom A disjoint from B, and (ii) implies that the probability of any atom A contained in B is $P_B(A)=1/m$, where m is the number of atoms in B. Now consider any event E and suppose k is the number of atoms contained in $E\cap B$, and N is the number of atoms in $\mathscr{E}$. Since E occurs if and only if $E\cap B$ occurs, we have

$$P_B(E)=\frac{k}{m}=\frac{k/N}{m/N}=\frac{P(E\cap B)}{P(B)}.$$

Hypothesis (iii) is the crucial assumption. We wish to compute P_B in terms of P, and (iii) is one of several possible ways of making this vague statement precise. The boolean algebra generated by E and B consists of all elements obtained by combining E and B in all possible ways using the operations $\cup$ (union), $\cap$ (intersection) and $'$ (complementation) (for example, $E\cup B'$, $E\cap B$ are two such combinations). There are sixteen events in this algebra. Hypothesis (iii) states that $P(E|B)$ is a function, the same for all probability spaces and measures, of the sixteen probabilities of these sixteen events. Furthermore, we assume this function of sixteen variables to be continuous. Is it reasonable to assume that $P(E|B)$ depends on $P(E\cap B')$, $P(E\cup B)$, $P(E'\cup B),\dots$ and only these? Hard to tell; it is certainly more reasonable than to assume directly that it equals $P(E\cap B)/P(B)$. Is it reasonable to assume that this dependence is continuous? Same answer. Without continuity the writer does not know whether the result can be established; it would be very interesting to see some weaker alternatives.

A simple lemma, whose proof we leave as an exercise, is useful in establishing our result.

LEMMA 1. *Given positive real numbers* q_i, $1 \leqslant i \leqslant n$, *such that* $\sum_{i=1}^{n} q_i = 1$ *and given* $\varepsilon > 0$, *there exist positive rational numbers* r_i, $1 \leqslant i \leqslant n$ *such that* $\sum_{i=1}^{n} r_i = 1$ *and for any subset* I *of* $\{1,2,\dots,n\}$ *we have* $|\sum_{i \in I} q_i - \sum_{i \in I} r_i| < \varepsilon$.

The next lemma is the key to our main result.

LEMMA 2. *Let* $(X,\mathscr{S})$ *be a finite probability space and* P *a probability measure on* $\mathscr{S}$. *Given* $\delta > 0$, *there exists a finite probability space* $(Y,\mathscr{F})$ *and a homomorphism* $\rho\colon \mathscr{S} \to \mathscr{F}$ *such that for all* $E \in \mathscr{S}$ *we have* $|P(E) - P_c(\rho E)| < \delta$, *where* P_c *is the counting probability measure on* $\mathscr{F}$.

Proof. Let $A_1, A_2, \dots, A_m$ be the atoms of $\mathscr{S}$, write $q_i = P(A_i)$ and suppose that $q_i > 0$ for $i = 1,2,\dots,n$, $q_i = 0$ for $i > n$. Now apply the lemma to obtain the positive rationals $r_1, r_2, \dots, r_n$. Next select positive integers k_i and k such that $r_i = k_i/k$, and pairwise disjoint sets E_i with cardinalities k_i, respectively ($i = 1,2,\dots,n$). Let $Y = \bigcup_{i=1}^{n} E_i$ and $\mathscr{F}$ the algebra of all subsets of Y. We are now ready to define our homomorphism ρ which is determined by its values on the atoms A_j of $\mathscr{S}$: we let $\rho(A_j) = E_j$ if $j \leqslant n$ and $\rho(A_j) = \varnothing$ if $j > n$. Since by construction $P_c(E_i) = r_i$ the lemma guarantees that $|P(E) - P_c(\rho E)| < \delta$ for any $E \in \mathscr{S}$.

We are now prepared to prove the main result.

THEOREM. *Suppose that hypotheses (i), (ii), and (iii) hold. Then, for any probability space* $(X,\mathscr{E})$, *any probability measure* P *on* $\mathscr{E}$ *and any event* $B \in \mathscr{E}$ *with* $P(B) \neq 0$, *we have*

$$P(E|B) = \frac{P(E \cap B)}{P(B)}.$$

Proof. Consider an event B with $P(B) \neq 0$, an event E, and fix them for the rest of the argument. Let $\mathscr{S}$ be the algebra generated by E and B (which we may assume distinct, for otherwise there is nothing to prove). The atoms of $\mathscr{S}$ are $E - B$, $B - E$, $E \cap B$ and $X - (E \cup B)$ and so the probabilities of the remaining events in the algebra can be continuously expressed in terms of $P(E-B)$, $P(B-E)$ and $P(E \cap B)$. Using hypothesis (iii) we then see that for some continuous function f mapping $[0,1] \times [0,1] \times [0,1] - \{(0,0,0)\}$ into $[0,1]$ we have

$$P(E|B) = f(P(E-B), P(B-E), P(E \cap B)).$$

Now take an $\varepsilon > 0$ and by continuity of the function f select a $\delta > 0$ such that

$$|f(P(E-B), P(B-E), P(E \cap B)) - f(x,y,z)| < \frac{\varepsilon}{2}$$

whenever $|P(E-B) - x|$, $|P(B-E) - y|$, and $|P(E \cap B) - z|$ are less than δ. Since $P(B) > 0$, we can restrict the size of this δ further to have $\delta < P(B)$ and $2\delta/P(B) < \varepsilon/2$. Now we observe that whatever the given algebra $\mathscr{E}$ is, the algebra $\mathscr{S}$ is finite; so we can apply the proposition we just proved using the δ we just selected to obtain the probability space $(Y,\mathscr{F})$ and the homomorphism $\rho\colon \mathscr{S} \to \mathscr{F}$ for which we have $|P(A) - P_c(\rho A)| < \delta$ for all $A \in \mathscr{S}$. Remember that P_c is the counting measure on $\mathscr{F}$ and we have shown that, by our hypotheses, conditioning of counting measures follows the classical formula. But our choice of δ gives $P_c(\rho B) > P(B) - \delta > 0$ so that we can condition with respect to ρB to obtain

$$\frac{P_c(\rho E \cap \rho B)}{P_c(\rho B)} = P_c(\rho E|\rho B) = f(P_c(\rho E - \rho B), P_c(\rho B - \rho E), P_c(\rho E \cap \rho B)).$$

On the other hand we have

$$\frac{P(E \cap B)}{P(B)} - \frac{P_c(\rho E \cap \rho B)}{P_c(\rho B)} = \frac{P(E \cap B) - P_c(\rho E \cap \rho B)}{P(B)} + \frac{P_c(\rho E \cap \rho B)}{P_c(\rho B)} \cdot \frac{P_c(\rho B) - P(B)}{P(B)};$$

since the second fraction is at most 1 and the numerators of the first and third are in absolute value at most δ, we obtain the inequality

$$\left|\frac{P(E\cap B)}{P(B)}-\frac{P_c(\rho E\cap \rho B)}{P_c(\rho B)}\right|<\frac{2\delta}{P(B)}<\frac{\varepsilon}{2}.$$

Finally note that

$$\begin{aligned}\left|P(E|B)-\frac{P(E\cap B)}{P(B)}\right| &= \left|f(P(E-B),P(B-E),P(E\cap B))-\frac{P(E\cap B)}{P(B)}\right|\\ &\leqslant |f(P(E-B),P(B-E),P(E\cap B))\\ &\quad -f(P_c(\rho E-\rho B),P_c(\rho B-\rho E),P_c(\rho E\cap \rho B))|\\ &\quad +\left|\frac{P_c(\rho E\cap \rho B)}{P_c(\rho B)}-\frac{P(E\cap B)}{P(B)}\right|;\end{aligned}$$

the first term is $<\varepsilon/2$ by the choice of δ and the second is $<\varepsilon/2$ as we just saw. Therefore we have

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for any $\varepsilon>0$.

There exist points of view differing from the classical Kolmogorov-type model of probability according to which probabilities of single events are meaningless and only conditional probabilities can be discussed. For this axiomatic treatment of conditioning, see A. Rényi **[1]** or, if the reader is interested in the original works, the papers by A. Rényi and Á. Császár [**2**] and [**3**]. The ideas of Rényi are extremely interesting, but are unrelated to the problem we have addressed. The results of Császár resemble ours, but are placed within the general axiomatic scheme of Rényi's work.

References

[1] A. Rényi, Foundations of Probability, Holden-Day Inc., 1970.

[2] ______, On a new axiomatic theory of probability, Acta Math. Acad. Sci. Hung., 6 (1955) 285–335.

[3] Á. Császár, Sur la structure des espaces de probabilite conditionelle, Acta Math. Acad. Sci. Hung., 6 (1955) 337–361.

A Certain Type of Number Expressible as the Sum of Two Squares

B. Suryanarayana Rao

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My attention was recently attracted to the numbers 5882353 and 99009901, both of which have the special property of being equal to the sum of the squares of their contiguous parts. Thus, $5882353=588\cdot 10^4+2353=588^2+2353^2$ and $99009901=990\cdot 10^5+09901=990^2+09901^2$.

since the second fraction is at most 1 and the numerators of the first and third are in absolute value at most δ, we obtain the inequality

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the first term is $<\varepsilon/2$ by the choice of δ and the second is $<\varepsilon/2$ as we just saw. Therefore we have

$$\left|P(E|B)-\frac{P(E\cap B)}{P(B)}\right|<\varepsilon$$

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There exist points of view differing from the classical Kolmogorov-type model of probability according to which probabilities of single events are meaningless and only conditional probabilities can be discussed. For this axiomatic treatment of conditioning, see A. Rényi [**1**] or, if the reader is interested in the original works, the papers by A. Rényi and Á. Császár [**2**] and [**3**]. The ideas of Rényi are extremely interesting, but are unrelated to the problem we have addressed. The results of Császár resemble ours, but are placed within the general axiomatic scheme of Rényi's work.

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These are instances of the Diophantine equation

$$au + b = a^2 + b^2 \tag{1}$$

where a, b, u are positive integers with $a < u$, $b < u$. If we think of $au + b$ as a number whose digits to the base u are a and b, we are asking in effect for two-digit numbers which are the sum of the squares of their digits. Now (1) is equivalent to

$$u^2 + 1 = (u - 2a)^2 + (2b - 1)^2 \tag{2}$$

so that the problem reduces to solving equation (2) for positive integers a, b for a given value of u. All we need to know are the ways in which $u^2 + 1$ can be written as the sum of two squares with the second square odd. Thus, let v and w be two positive integers for which $u^2 + 1 = v^2 + w^2$, w is odd and u and v have the same parity. Four solutions to (2) can be found by setting $u - 2a = \pm v$, $2b - 1 = \pm w$. These are

$$(a, b) = \left(\tfrac{1}{2}(u - w), \tfrac{1}{2}(w + 1)\right), \ \left(\tfrac{1}{2}(u + v), \tfrac{1}{2}(w + 1)\right),$$
$$\left(\tfrac{1}{2}(u - v), -\tfrac{1}{2}(w - 1)\right), \left(\tfrac{1}{2}(u + v), -\tfrac{1}{2}(w - 1)\right);$$

the first two yield positive values of b. For example, when $u = 13$, then $u^2 + 1 = 1^2 + 13^2 = 13^2 + 1^2 = 7^2 + 11^2 = 11^2 + 7^2$ generates the solutions $(6,7), (7,7), (6,-6), (7,-6)$; $(0,1), (13,1), (0,0), (13,0)$; $(3,6), (10,6), (3,-5), (10,-5)$; $(1,4), (12,4), (1,-3), (12,-3)$. For $u = 10^2$, we can use the representation $10^4 + 1 = 76^2 + 65^2$ to generate the numbers $1233 = 12 \cdot 10^2 + 33 = 12^2 + 33^2$ and $8833 = 88 \cdot 10^2 + 33 = 88^2 + 33^2$ with the property mentioned at the beginning of this note. If we adopt a bar notation analogous to that for logarithms, the other two solutions corresponding to this representation yield the numbers $12\overline{32} = 1168 = 12^2 + 32^2$ and $88\overline{32} = 8768 = 88^2 + 32^2$.

There is a second way in which the number $u^2 + 1$ can be used to generate numbers with the special property. Suppose that we can write it as a product of two factors, $u^2 + 1 = (x^2 + y^2)(z^2 + w^2)$, with the additional condition $xz - yw = 1$. Then $(xw + yz)^2 + 1 = (xw + yz)^2 + (xz - yw)^2 = u^2 + 1$, so that $xw + yz = u$. It is straightforward to check that equation (1) is satisfied by $(a, b) = (xw, xz), (yz, xz), (xw, -yw), (yz, -yw)$. Let us examine some examples:

EXAMPLE 1. $u = 13$, $u^2 + 1 = 170 = (1^2 + 0^2)(1^2 + 13^2) = (1^2 + 1^2)(7^2 + 6^2) = (1^2 + 3^2)(4^2 + 1^2) = (3^2 + 5^2)(2^2 + 1^2)$. The last representation yields the numbers $45 = 3 \cdot 13 + 6 = 3^2 + 6^2$, $136 = 10 \cdot 13 + 6 = 10^2 + 6^2$, $34 = 3 \cdot 13 - 5 = 3^2 + 5^2$ and $125 = 10 \cdot 13 - 5 = 10^2 + 5^2$, corresponding to the solutions $(a, b) = (3,6)$, $(10,6)$, $(3,-5)$ and $(10,-5)$. The other representations can be handled similarly.

EXAMPLE 2. $u = 10^2$, $u^2 + 1 = 73 \cdot 137 = (3^2 + 8^2)(11^2 + 4^2)$, which yields the numbers $1233, 8833, 1168 = 12\overline{32}$ and $8768 = 88\overline{32}$.

EXAMPLE 3. $u = 10^4$. Using $10^2 \equiv -2 \pmod{17}$, it is easy to check that $u^2 + 1$ is divisible by 17. Writing $10^8 + 1 = (1^2 + 4^2)(p^2 + q^2)$ with $p - 4q = 1$ and $4p + q = 10^4$, we find that $p = 2353$ and $q = 588$. This yields the numbers 5882353 and $94122353 = 9412^2 + 2353^2$.

EXAMPLE 4. $u = 10^5$, $u^2 + 1 = (1^2 + 10^2)(9901^2 + 990^2)$. This generates 99009901 and $9901009901 = 99010^2 + 09901^2$.

EXAMPLE 5. $u = 10^8$, $u^2 + 1 = 10^{16} + 1$. Now $6 \cdot 10^3 + 1 = 353 \cdot 17$ and $6^5 = 10 + 353 \cdot 22$, so that $6(10^{16} + 1) \equiv 10(6 \cdot 10^3)^5 + 6^5 \equiv 10(-1) + 6^5 \equiv 0 \pmod{353}$. Letting $10^{16} + 1 = 353(p^2 + q^2) = (17^2 + 8^2)(p^2 + q^2)$ with $17p - 8q = 1$ and $8p + 17q = 10^8$, we find that $p = 2266289$ and $q = 4815864$, from which solutions to (1) can be generated.

PROBLEMS

LEROY F. MEYERS, Editor
G. A. EDGAR, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before February 1, 1985.

1196. (a) Prove that the area of a triangle whose vertices are integer lattice points in the plane is always half an (even or odd) integer.

(b) Suppose three lattice points are chosen at random. What is the probability that the area of the triangle they determine is an integer? More precisely, if the points are chosen from a large rectangle, does the probability that the area is an integer converge as the dimensions of the rectangle grow without bound? [*Eric C. Nummela, New England College.*]

1197. Characterize the triangles of which the midpoints of the altitudes are collinear. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

1198. Prove the identity

$$\sum_{k=0}^{m-1} \csc^2\left(\frac{k\pi}{m}+\alpha\right) = m^2 \csc^2(m\alpha),$$

provided that $m\alpha$ is not an integral multiple of π. [*Russell Euler, Northwest Missouri State University.*]

1199. In the isosceles triangle ABC, with $AB = AC$, let H be the foot of the altitude from A, let E be the foot of the perpendicular from H to AB, and let M be the midpoint of EH. Show that $AM \perp EC$. [*Aristomenis Siskakis, University of Illinois.*]

1200. Let K_n be the complete graph on n vertices, and G any subgraph of K_n having at most $n-3$ edges. Then K_n has a Hamiltonian circuit not containing any edge of G. [*Paul Erdős, Hungarian Academy of Sciences.*]

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*We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) will be placed next to a problem number to indicate that the proposer did not supply a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address. It is not necessary to submit duplicate copies.

Send all communications to the problems department to Leroy F. Meyers, Mathematics Department, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.

Quickies

Solutions to Quickies appear on p. 242.

Q691. Liouville's theorem states that any non-constant analytic function becomes unbounded in the complex plane. Show that for any region of the complex plane where the cosine function is unbounded, the sine function is similarly unbounded. [*Frank W. Schmidt and Rodica Simion, Southern Illinois University.*]

Q692. Let a and b be integers. Show that if $\frac{1}{2}(a^3+b^3)$ is prime, then so are $3a^2-6a+4$ and $3b^2-6b+4$. [*Bill Olk, student, Carroll College, Wisconsin.*]

Q693. Let p be prime, and let M be a set of p consecutive positive integers. Is it possible to partition M into two sets M_1 and M_2 (where $M_1 \cup M_2 = M$ and $M_1 \cap M_2 = \varnothing$) such that the product of the numbers in M_1 is equal to the product of the numbers in M_2? [*Florentin Smarandache, Lycée Sidi El Hassan Lyoussi, Sefrou, Morocco.*]

Q694. $ACEG$ is a quadrilateral circumscribed about a circle and tangent to the circle at the points B, D, F, and H, with B lying on AC, D lying on CE, F lying on EG, and H lying on GA. If $AB=3$, $CD=4$, $EF=5$, and $GH=6$, find the radius of the circle. [*John P. Hoyt, Lancaster, Pennsylvania.*]

Solutions

Divisibility of a Combinatorial Sum **September 1983**

1175. Suppose that $m=nq$, where n and q are positive integers. Prove that the sum of binomial coefficients

$$\sum_{k=0}^{n-1}\binom{(n,k)q}{(n,k)}$$

is divisible by m, where (x,y) denotes the greatest common divisor of x and y. [*Anon, Erewhon-upon-Spanish River.*]

Solution I: Let $G=\mathbf{Z}/(m)$ and let X denote the set of all n-element subsets of G. Then G acts on X by translation. Under the action of G, the set X splits into orbits (transitivity classes). By a well known formula of Burnside, the number of orbits is

$$\frac{1}{|G|}\sum_{a\in G}\psi(a),$$

where for each a in G the number $\psi(a)$ denotes the number of sets S such that $a+S=S$. Now the order of a is $m/(m,a)$. It is easily seen that an n-set S is invariant under a if and only if S

decomposes into orbits of single elements under the $m/(m,a)$ element subgroup $\langle a \rangle$. This means that $\psi(a) > 0$ if and only if $m/(m,a)$ divides n. A short calculation shows that this is the case if and only if q divides a; hence $\psi(a) > 0$ if and only if $a = qk$ for some k such that $0 \leqq k \leqq n-1$. Then

$$\operatorname{ord} a = \frac{m}{(m,a)} = \frac{nq}{(nq,qk)} = \frac{n}{(n,k)}.$$

Thus G breaks up into

$$\frac{m}{\operatorname{ord} a} = \frac{nq}{n/(n,k)} = (n,k)q$$

orbits under $\langle a \rangle$, and we must choose

$$\frac{n}{\operatorname{ord} a} = \frac{n}{n/(n,k)} = (n,k)$$

of them to make a typical n-element set S such that $a + S = S$. Thus if $a = qk$, then

$$\psi(a) = \binom{(n,k)q}{(n,k)}.$$

From the Burnside formula, the sum of the $\psi(qk)$ is divisible by $|G| = m$.

ANON
Erewhon-upon-Spanish River

Solution II will appear in the November issue.

The Adjoint Matrix's Characteristic Polynomial **September 1983**

1176. Let $f(t) = \det(tI - A)$ be the characteristic polynomial of the $n \times n$ matrix A. Find a formula for the characteristic polynomial of the adjoint of A in terms of f. (The adjoint of A, denoted $\operatorname{adj} A$, is the $n \times n$ matrix whose (r,s) element is the (s,r) cofactor of A.) [*H. Kestelman, University College London, England.*]

Solution: For $n \geqq 2$, consider the field of quotients of polynomials in the $n^2 + 1$ commuting independent indeterminates x_{ij} $(i,j = 1,\ldots,n)$ and t over some field F, and let $A = (x_{ij})$. Let A' denote the (classical) adjoint of A, let $y = \det A$, and let $f(t) = \det(tI - A) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, where the a_k are polynomials in the x_{ij} and $a_0 = (-1)^n y$. Then the characteristic polynomial $g(t) = \det(tI - A')$ of A' is shown to be the polynomial in t and the x_{ij} given by

$$g(t) = t^n + (-1)^n\left(a_1 t^{n-1} + a_2 y t^{n-2} + \cdots + a_{n-1} y^{n-2} t + y^{n-1}\right). \tag{1}$$

Since $AA' = yI$ and $\det A' = y^{n-1}$, we have

$$\det(tI - A') = \det((t/y)AA' - A') = (-1)^n (t/y)^n \det((y/t)I - A)\det A'$$
$$= (-1)^n t^n f(y/t)/y = t^n + (-1)^n\left(a_1 t^{n-1} + \cdots + y^{n-1}\right),$$

as asserted, by the usual properties of determinants.

Observe that (1) remains valid if elements a_{ij} of F are substituted for the x_{ij}. Further, if rank $(a_{ij}) < n-1$, then $g(t)$ is simply t^n, while if $\operatorname{rank}(a_{ij}) = n-1$, then $g(t) = t^n + (-1)^n a_1 t^{n-1}$, where $a_1 \neq 0$. Finally, note that if $n = 2$, then $g(t) = f(t)$.

LORRAINE L. FOSTER
California State University, Northridge

Also solved by P. J. Anderson (Canada), Irl C. Bivens, Thomas E. Elsner, Konrad J. Heuvers, Vania D. Mascioni (student, Switzerland), William A. Newcomb, Richard Parris, and the proposer; and partially (non-singular case only) by David J. Boduch (student), Hans Kappus (Switzerland), David Lindsay, and Gordon Williams.

Mascioni and Newcomb derived the formula

$$g(t)=t^n+(-1)^n f'(0)t^{n-1}+\sum_{k=2}^{n}(-1)^{nk}f^{(k)}(0)(f(0))^{k-1}t^{n-k}/k!,$$

valid in all cases.

Finite Fields, Matrices, an Euler Product, and Irrational Numbers **September 1983**

1177. Let $f_q(n)$ denote the probability that an $n\times n$ matrix with entries chosen at random from $\mathrm{GF}(q)$, the finite field of q elements, is invertible over $\mathrm{GF}(q)$.

(a) Show that $\lim_{n\to\infty}f_q(n)=\prod_{k=1}^{\infty}(1-q^{-k})$.

(b) Show that the value of the infinite product in (a) is irrational for every integer $q\geqslant 2$. [*Manuel Blum and J. O. Shallit, University of California.*]

Solution: (a) An $n\times n$ matrix A over $\mathrm{GF}(q)$ is invertible iff the row vectors $R_1, R_2,\dots, R_n$ of A are linearly independent. There are q^n $1\times n$ row vectors over $\mathrm{GF}(q)$. Also, there are q^k linear combinations of k linearly independent row vectors over $\mathrm{GF}(q)$. Thus the probability that $R_1, R_2,\dots, R_{k+1}$ are linearly independent, given that $R_1, R_2,\dots, R_k$ are linearly independent, is $(q^n-q^k)/q^n$, if $1\leqq k\leqq n-1$. Since $R_1\neq \mathbf{0}$ with probability $(q^n-1)/q^n$, it is clear that

$$f_q(n)=\prod_{k=0}^{n-1}(1-q^{k-n})=\prod_{k=1}^{n}(1-q^{-k}). \tag{1}$$

(b) By an identity due to Euler, product (1) converges to

$$H(q)=1+\sum_{k=1}^{\infty}(-1)^k\left(q^{-k(3k-1)/2}+q^{-k(3k+1)/2}\right).$$

(See for example Hardy and Wright, *An Introduction to the Theory of Numbers*, 1960, p. 284.) We may rewrite the summation as

$$(1-q^{-1}-q^{-2})+\sum_{m=1}^{\infty}\left[q^{-m(6m-1)}+\left(q^{-m(6m+1)}-q^{-(2m+1)(3m+1)}-q^{-(2m+1)(3m+2)}\right)\right],$$

where the quantities in parentheses are positive. Thus in the representation $0.b_1b_2b_3\dots$ of $H(q)$ in base q there are arbitrarily long strings of consecutive zeros. In fact, $b_{m(6m-1)}=1$, whereas $b_j=0$ for $m(6m-1)<j\leqq m(6m+1)$, if $m\geqq 1$. Hence $\lim_{n\to\infty}f_n(q)$ is irrational.

LORRAINE L. FOSTER
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Also solved by Chico Problem Group, Richard Parris (part (a) only), and the proposers.

Answers

Solutions to the Quickies on p. 239.

Q691. The result follows immediately from the identity $\sin^2 z + \cos^2 z = 1$, which remains valid for complex z.

Q692. Since $p = (a^3 + b^3)/2$ is a positive integer not divisible by 4, a and b must be odd. Hence $(a+b)/2$ is a positive integer m. Then $p = m(3a^2 - 6ma + 4m^2) = m(3(m-a)^2 + m^2)$. If $m \neq 1$, then from $0 \leqq 3(m-a)^2 + m^2 \neq 1$ it would follow that p is not prime. Hence $m = 1$, and so $3a^2 - 6a + 4$ equals p and the value of $3b^2 - 6b + 4$ is the same.

Q693. Exactly one of the numbers in M is divisible by p. Hence the subset containing the number has a product which is divisible by p, whereas the other subset's product is not divisible by p. Hence no such partition is possible.

[*Editor's note.* The problem is equivalent to asking if the product of p consecutive positive integers can be a square. For references to early work on this and related problems, see L. E. Dickson, *History of the Theory of Numbers*, vol. II, pp. 679–680. For later work see P. Erdős, "Note on products of consecutive integers", *J. London Math. Soc.*, v. 14 (1939) 194–198 and 245–249; *Indag. Math.*, v. 17 (1955) 85–90. See also *Crux Mathematicorum* problem 83: v. 2 (1976) 28–29; v. 9 (1983) 278.]

Q694. If $e = AB$, $f = CD$, $g = EF$, $h = GH$, and O and r are the center and radius of the circle, then $\angle OAB + \angle OCD + \angle OEF + \angle OGH = \pi$. Hence

$$\cot(\angle OAB + \angle OCD) = -\cot(\angle OEF + \angle OGH),$$

or

$$\frac{\frac{e}{r}\cdot\frac{f}{r} - 1}{\frac{e}{r} + \frac{f}{r}} = -\frac{\frac{g}{r}\cdot\frac{h}{r} - 1}{\frac{g}{r} + \frac{h}{r}},$$

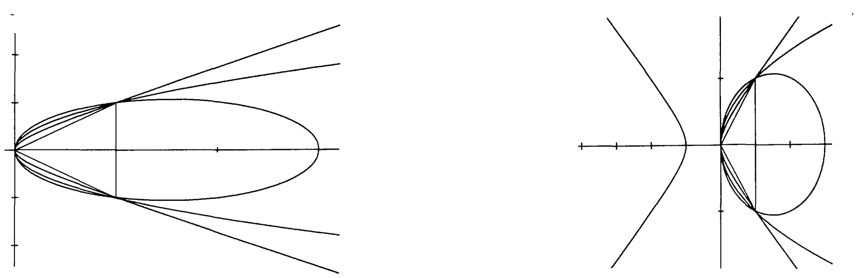
which reduces to

$$r^2 = \frac{efg + fgh + ghe + hef}{e + f + g + h}.$$

The given measurements yield $r = \sqrt{19}$.

[*Ed. note*: Nine additional problems in Euclidean geometry may be found in the proposer's "A short test for geometry teachers", *School Science and Mathematics*, v. 41 (1941), 384.]

Outtakes



Two more designs by Gary Eichelsdorfer which illustrate Archimedes generalized (p. 224).

REVIEWS

PAUL J. CAMPBELL, Editor
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

te Riele, Herman, *Mertens' conjecture disproved, CWI [Centrum voor Wiskunde an Informat]* Newsletter 1:1 (January, 1984) 23-24.

F. Mertens offered a simple number-theoretic conjecture in 1897 whose truth would imply the Riemann Hypothesis. The author (CWI) and Andrew Odlyzko (Bell Labs) used a computer to show the conjecture is false. (The conjecture: $M(x) < \sqrt{x}$ for all $x > 1$, where $M(x)$ is the sum over all $n \leq x$ of $\mu(n)$, μ being the Möbius function. The fact: lim sup $(M(x)/\sqrt{x}) \geq \bar{1}.06$).

Kolata, Gina, *Order out of chaos in computers*, Science 223 (2 March 1984) 917-919.

Several notorious problems in computer science, for which deterministic algorithms are unknown or extremely complicated, have yielded to simple probabilistic algorithms. The randomized solution by M. Rabin and D. Lehmann of Dijkstra's dining philosophers problem avoids deadlock and starvation "with probability 1"; and Rabin's solution to the Byzantine generals problem is akin to probabilistic primality testing, in that successive iterations of the algorithm reduce the probability of failure by 1/2 each. The problems in question have utterly serious analogues in the coordination and synchronization of computing units which are subject to failure. How secure the public will feel about computers using probabilistic approaches remains to be seen, however, especially since most people believe computers already make enough mistakes without being "allowed" the "chance" to make more.

Davies, Paul, *The eleven dimensions of reality*, New Scientist 101 (9 February 1984) 31-33.

"An age-old scientific dream is that the physical world might be reducible, ultimately, to pure geometry ..." Few dream that the geometry should involve 11 dimensions! Two years ago, New Scientist published an article on "Living in a five-dimensional world" (29 April 1982, p. 296); that world, the Kaluza-Klein theory, grafts one extra dimension onto space-time, to incorporate electromagnetism. Inclusion of the weak and strong forces too demands still more dimensions, for a total of 11. Curiously, the article hints that Milnor's 1956 discovery of "wild spheres" in 7-dimensional space may have provided a key ingredient, since the seven unperceived dimensions must make each perceived point a "slightly distorted" microscopic 7-sphere--and the geometric properties are precisely what are required if the extra dimensions are to be interpreted as force fields. In fact, the most elegant formulation of a supersymmetric theory of gravity requires exactly 11 dimensions!

Kolata, Gina, *Graph theory result proved,* Science 224 (4 May 1984) 480-481.

In 1930 Kuratowski discovered the two graphs which are "minimal graphs" for the plane: every nonplanar graph must contain either a complete graph on five vertices (K_5) or else a utilities graph ($K_{3,3}$). Fifty years later, Archdeacon, Huneke, and others found that there are 103 minimal graphs for the Möbius strip ("minimal" in the sense that none contains a copy of any of the others). For the torus there are at least 800, and for a sphere with two handles at least 80000. Seymour and Robinson have just provided some relief by showing that the list of minimal graphs is finite for any surface. But perhaps in addition an astonishingly stronger 20-year-old conjecture of Wagner's may be true: that *any* list of graphs with no graph contained in any other must be finite! Robertson and Seymour have partial results, including a proof for any list that contains a planar graph.

Emerson, John D., and Colditz, Graham A., *Statistics in practice: use of statistical analyses in the New England Journal of Medicine,* New England Journal of Medicine 309 (1983) 709-713.

"Will knowledge of a few elementary statistical techniques, such as chi-square and t-test analyses, assist readers in understanding the statistical component of a high percentage of research articles in the *Journal*?" The authors found that a reader conversant with descriptive statistics has complete access to 58% of the articles, understanding t-tests increases this access to 67%, and adding contingency tables pushes it to 73%. Other topics common in an introductory statistics course (non-parametric tests, correlation, sample regression, analysis of variance, data transformations, non-parametric correlation, and multiple regression) bring the access to 86%.

Bust of Ramanujan Unveiled, The Hindu (Madras) (12 May 1984) 9.

It may surprise more than a few readers that the widow of Srinivasa Ramanujan is still living, more than 65 years after the famous mathematician's death. Her desire to see him memorialized in India by a statue led R. Askey (Wisconsin) to commission a bronze bust and organize a subscription campaign (still open) to which more than 100 mathematicians and scientists have contributed. (A sidebar notes possible confusion over Ramanujan's year of birth, most likely occasioned by misreading a numeral in one of his letters; the centenary of his birth will be celebrated in 1987.)

Hopcroft, John E., *Turing machines,* Scientific American 250:5 (May 1984) 86-98, 154.

What does it mean for a function to be computable? In 1936 Alan Turing and others formulated answers, and Turing's was in terms of the idealized machine that now bears his name. The article describes in detail how the machine can do calculation, and it also gives an example of an uncomputable function (the "busy beaver" function) plus a brief resumé of complexity theory.

Greenwell, Raymond N., *Microcomputers in undergraduate mathematics: writing the successful program,* Collegiate Microcomputer 1 (May 1983) 119-123.

"...[M]any microcomputer programs are of dubious value because they do nothing that a good text or a good blackboard presentation could not do." The author notes the motivating power of graphics, animation, and sound; and he emphasizes that programs have to be fast. He includes a program (for the Apple) that graphs the direction field of a differential equation; the user then chooses an initial point by using the paddles, and the program draws in the solution curve through that point. No doubt more adulation would accrue to a teacher using a rewritten version employing the "magic wand" of a light pen!

Byrkett, Donald L., *Using computers to enhance the content of undergraduate mathematics courses*, Collegiate Microcomputer 1 (1983) 365-371.

"...[C]are must be taken to use the computer properly in mathematics classes so that the primary emphasis is on the mathematical principles and the analysis of problems and not on the programming or computer operation details... . This paper offers several suggestions for the *appropriate* use of computers as an experimental tool, as an algorithmic tool, and as a tool to solve realistic size problems using commercial software."

Watson, Andrew, *The search for supersymmetry*, New Scientist (15 March 1984) 28-30.

In 1918 Emmy Noether published a theorem that physicists are now trying to exploit in modeling the dynamics of interacting elementary particles: symmetries in the mathematics correspond to conservation laws. Supersymmetry is a new symmetry principle linking fermions (whose spin is 1/2) and bosons (whose spin is 0 or 1). Not yet confirmed by experiments, supersymmetry if exact would predict a spin 0 partner to the electron with the same mass. The theory, however, does handle nicely some of the thorny issues that plague grand unified theories.

Bürger, Wolfgang, *The yo-yo: a toy flywheel*, American Scientist (March-April 1984) 137-142.

I always wondered why I couldn't do all those marvelous yo-yo tricks. Now I know: I had a "classical" yo-yo, not a "modern" one! This article describes in quantitative terms (an elliptic integral makes a brief appearance) the behavior of both, and also of the "academic" yo-yo, whose string has negligible thickness.

Miller, Julie Ann, *Mendel's peas: a matter of genius or of guile*, Science News 125 (18 February 1984) 108-109.

Poor Mendel! Ronald Fischer in 1936 concluded that Mendel's observations were too good to be true. A recent paper in History of Science (21 (1983) 275ff) by R. S. Root-Bernstein suggests that Mendel merely used a "subjective, theory-directed counting procedure," assigning the difficult-to-classify peas in a way that maximized agreement with the theory. M. Usselman agrees, but asserts "Selection and interpretation are what makes a scientist great." (It is also what makes a fiction writer great, as Robert Penn Warren once noted.) Should students today believe Mendel's results? In the true spirit of science they should try to replicate his experiments, and try classifying peas by characters--as Root-Bernstein had undergraduates do.

Beckwith, Jonathan, and Woodruff, Michael, *Letters: Achievement in mathematics*, Science 223 (23 March 1984) 1247-1248.

A microbiologist and a biochemist criticize the report by Benbow and Stanley (2 December 1983, p. 1029) on difference in performance of mathematically precocious boys and girls by questioning the significance of SAT-M scores as measures of mathematical ability or predictors for entrance and success in a mathematical career. They also note the study fails to look into socialization factors.

Skolnick, Joan, *et al.*, How to Encourage Girls in Math & Science: Strategies for Parents and Educators, Prentice-Hall, 1982; xi + 192 pp, (P).

A practical, informative book that both details the process of sex-role socialization and provides specific strategies and activities to encourage learning of math and science. Aimed at girls through the eighth grade, the book offers a wealth of material to improve their problem-solving skills and build confidence in their abilities.

New Goals for Mathematical Sciences Education: The Report of a Conference Sponsored by the Conference Board of the Mathematical Sciences, November 13-15, 1983 (single copy free for self-addressed mailing label to "New Goals, CBMS," at MAA headquarters), iv + 22 pp.

A response by members of mathematical sciences organizations to the flood of recent reports documenting the widespread need for improvement in mathematical sciences education in the U.S. The recommendations include establishment of a continuing task force on curriculum; establishment of local teacher support networks to link colleagues at every level; use of early "prognostic" testing to allow for remediation before college; strong efforts to increase public awareness of the importance of mathematics and of identification and encouragement of the able and gifted (particularly among women and minorities); new efforts and approaches to remediation; and new initiatives to renew mathematics teachers' knowledge, skills, and enthusiasm.

Pottage, John, <u>Geometrical Investigations Illustrating the Art of Discovery in the Mathematical Field</u>, Addison-Wesley, 1983; xxi + 480 pp.

"... a Dialogue in the Galilean Style in which Salviati, Sagredo, and Simplicio, Being Returned from the Shades to Discuss a Problem Concerning PERIMETER, AREA, and VOLUME RATIOS and Thereby Led to Examine in an Elementary Fashion the RECTIFICATION of the ELLIPSE and the Nature of OVALS OF UNIFORM BREADTH and to Touch upon the PROBLEM OF ISOPERIMETRY ... . With the addition of Numerous Annotations and Problems by the Author ...". A splendid addition to the scant literature of heuristic that may appeal to non-mathematicians.

Abraham, Ralph H., and Shaw, Christopher D., <u>Dynamics--The Geometry of Behavior. Part 1: Periodic Behavior</u>, Aerial Pr (Box 1360, Santa Cruz, CA 95061); x + 220 pp, $29 (P); Abraham, Ralph, and Norkog, Lance, Disk One: <u>Periodic Attractors in the Plane</u>, $15 (Apple disk to accompany the book).

This amazing and beautiful introduction to non-linear dynamics is couched purely in visual and geometric terms; the corresponding equations appear only in the appendix. Attractors, limit cycles, and invariant tori: all arise from the explanations of the common experience of pendula, buckling columns, musical instruments, predator-prey systems, and forced oscillations. Four-color diagrams take up 90% of the book, apart from a 10-page "Dynamics Hall of Fame" of portraits and biographical sketches. This book is extremely valuable in conveying intuitive understanding of these phenomena. The accompanying software allows users to vary parameters in the different systems and observe corresponding vector fields and phase portraits. (The programs do run painfully slowly.) The combined package is part one of an announced four-package series.

Axelrod, Robert, <u>The Evolution of Cooperation</u>, Basic Books, 1984; x + 241 pp, $17.95.

"Under what conditions will cooperation emerge in a world of egoists without central authority?" The author combines empirical research (on strategies for Prisoner's Dilemma) and historical perspective (the live-and-let-live system in WWI trench warfare), plus biological analogy and mathematical theorems, to lead to the conclusion: "What makes it possible for cooperation to emerge is the fact that the players may meet again ... . The foundation of cooperation is not really trust, but the durability of the relationship." The curious "success" of the Tit-for-Tat strategy in Prisoner's Dilemma in the author's tournament leads him to remarkable advice on how to promote cooperation. This is a fascinating and important book, a first-rate example of the application of mathematics to human affairs.

Meyer, Walter J., Concepts of Mathematical Modeling, McGraw-Hill 1984; xi + 440 pp.

Most books on mathematical modeling are organized by mathematical techniques or by area of application. This book offers an approach featuring aspects of the modeling *process*: strategies, attitudes toward data, choices to be made. The level is elementary enough (calculus, a little probability and matrix theory) to allow students to study modeling in the sophomore year; the subject matter is delightfully varied and difference equations and optimization are treated. Too many problems are unoriginal (sources *are* credited) and/or smack of the unreal. If modeling is important, why wait till the senior year?

Gnanadesikan, Ram, Statistical Data Analysis, Amer. Math. Soc., 1983; ix + 141 pp.

This is a well-organized compilation of papers on which a 1982 AMS Short Course was based. Topics treated include graphical methods, robust methods, multilinear methods, and a case study using factor analysis and analysis of variance.

Kahneman, David, *et al.* (eds.), Judgment Under Uncertainty: Heuristics and Biases, Cambridge U Pr, 1982; xii + 555 pp.

Based on research of the last 10 years, "We now know that people's intuitive inferences, predictions, probability assessments, and diagnoses do not conform to the laws of probability theory and statistics." The resulting judgmental biases can be "large, persistent and serious in their implications for decision making"; the article by Eddy on physicians' intuitive probabilities is scary evidence for this assertion. Other authors in this volume identify and focus on the heuristics people use in place of probability calculus.

Henderson, Linda D., The Fourth Dimension and Non-Euclidean Geometry in Modern Art, Princeton U Pr, 1983; xxiii + 453 pp + 116 plates, $60, $18.50 (P).

The first three decades of this century were an intense period in art, and the author finds that the concept of the fourth dimension was a concern in every major movement. That is, a fourth *spatial* dimension--for Henderson finds that the new concepts of space in art are not to be traced to the influence of relativity theory and its fourth dimension of time. Nor did noneuclidean geometry never achieve the same widespread popularity as the fourth dimension, "which possessed many nongeometric associations." She details the threads of those associations in the modern art of Europe and America, and her work fills a void in the history of mathematics interacting with other elements of culture.

Bentley, Jon Louis, Writing Efficient Programs, Prentice-Hall, 1982; xvi + 170 pp, (P).

Excellent and clear advice from the columnist who writes "Programming Pearls" for the Communications of the Association for Computing Machinery. He formulates rules for speeding up a computation, and does so in the context of other important software values such as maintainability, robustness, and development time. Good design, layout and typography nicely complement the ideas; the book is destined to become a classic.

Frey, Peter W. (ed.), Chess Skill in Man and Machine, 2nd ed., Springer-Verlag, 1983; xiv + 329 pp.

Since this book first appeared in 1977, developments in computing have led to machines playing at the master level (USCF rating 2200-2400). This new edition includes a summary of recent games and two new chapters: on Belle, the current world champion and the most effective search-intensive program; and on Paradise, the most impressive example of a knowledge-intensive program.

NEWS & LETTERS

ALLENDOERFER, FORD AND PÓLYA 1983 AWARDS

At the business meeting on August 17, 1984, in Eugene, Oregon, the MAA honored eight authors for excellence in expository writing. The awards, in the amount of $200 each, were for articles published in 1983 in *Mathematics Magazine*, the *American Mathematical Monthly*, and the *Two-Year College Mathematics Journal*.

Recipient of the Carl B. Allendoerfer award was:

Judith V. Grabiner, "The Changing Concept of Change: The Derivative from Fermat to Weierstrass", *Math. Magazine*, 56 (1983) 195-206.

Recipients of the Lester R. Ford awards were:

Judith V. Grabiner, "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus", *Amer. Math. Monthly*, 90 (1983) 185-194.

Roger Howe, "Very Basic Lie Theory", *Amer. Math. Monthly*, 90 (1983) 600-623.

John Milnor, "On the Geometry of the Kepler Problem", *Amer. Math. Monthly*, 90 (1983) 353-365.

Joel Spencer, "Large Numbers and Unprovable Theorems", *Amer. Math. Monthly*, 90 (1983) 669-675.

William C. Waterhouse, "Do Symmetric Problems Have Symmetric Solutions?", *Amer. Math. Monthly*, 90 (1983) 378-387.

Recipients of the George Pólya awards were:

Ruma Falk and Maya Bar-Hillel, "Probabilistic Dependence Between Events", *TYCMJ*, 14 (1983) 240-247.

Richard J. Trudeau, "How Big is a Point?", *TYCMJ*, 14 (1983) 295-300.

1987 YEARBOOK WRITERS SOUGHT

The Educational Materials Committee of the National Council of Teachers of Mathematics invites articles for the 1987 NCTM Yearbook, Geometry Today. This yearbook, edited by Mary Montgomery Lindquist of Columbus (GA) College, deals with geometry at all levels of the K-12 mathematics curriculum. Proposals must be received by 1 March 1985. Guidelines for writers, including instructions for preparing proposals, are available from Albert P. Shulte, General Yearbook Editor, Oakland Schools, 2100 Pontiac Lake Road, Pontiac, MI 48054. Suggestions of persons who should be encouraged to submit manuscripts should also be sent to the General Yearbook Editor.

LECTURER PROGRAM IN STATISTICS

The COPSS Visiting Lecturer Program in Statistics is continuing into its 22nd successive year. It is sponsored by the Committee of Presidents of Statistical Societies and is supported jointly by the principal statistical organizations in North America: the American Statistical Association, the Biometric Society, the Institute of Mathematical Statistics, the Society of Actuaries, and the Statistical Society of Canada. Leading statisticians -- from universities, industry, and government -- have agreed to participate as lecturers. Lecture topics include subjects in experimental and theoretical statistics, as well as probability, information theory and stochastic models in the physical, biological, and social sciences.

The program is available to colleges, high schools, and other interested groups in the continental U.S.A. and Canada.

The purpose of the program is to provide information to high school and college students and faculties about the nature and scope of modern statistics, advice for undergraduate and graduate study, college curricula, and careers in statistics.

A detailed brochure describing the program is available from: Jon R. Kettenring, Chair, Visiting Lecturer Program in Statistics, 29 Pine Grove Avenue, Summit, NJ 07901.

AWM SPEAKERS' BUREAU

The Association for Women in Mathematics sponsors a Speakers' Bureau, currently funded by the Alfred P. Sloan Foundation, which offers women mathematicians as lecturers in schools, colleges and universities. A brochure for the Bureau contains a list of speakers, their sample topics and suggested audience levels of presentation. The brochure also contains the names, addresses and telephone numbers of the regional and metropolitan coordinators. Honoraria and travelling expenses, up to a certain limit, are reimbursed by the AWM from the Sloan grant. For brochure and further information about the Speakers' Bureau write or telephone the AWM Office, Box 178, Wellesley College, Wellesley, MA 02181, (617) 235-0320, Ext. 2643.

FOLLOWUP ON ELEVATORS

In "A second look at a theory of elevators," in International Journal of Mathematical Education in Science and Technology 15:1 (1984) 121-125, A. M. Vaidya relates a classroom experience of exploring further the model of A Wuffle in this *Magazine* 55 (1982) 30-37, 187.

-- Paul Campbell
Reviews Editor

BRIEF SOLUTION TO PUTNAM A-2

Problem A-2 in the 1983 Putnam competition reads: The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

Here is a third solution (see this *Magazine*, May 1984, 187-188 for Solutions I and II) using neither calculus nor vectors nor trigonometry.

Sol. III. Assume the *long* hand is fixed in the (fixed) plane of the clock face and let OP,OQ denote the short and long hands respectively. Thus Q is stationary and P moves with constant speed. Hence the distance QP is increasing most rapidly when P is moving directly away from Q -- i.e. when QP is perpendicular to OP.

This solution presents an interesting paradox: the first idea which springs to mind is to fix the *slow*-moving hand (as in *Sol. II*), but this does not lead to any solution comparable to that obtained from the less obvious idea of fixing the fast-moving hand, since there is no instant when Q is moving directly away from P.

-- J. G. Mauldon
Amherst College

Mauldon's solution, though found independently, is essentially the same as the Putnam committee's solution. -- ed.

A CALCULUS SHORTCUT

Problem 2 of the 12th USAMO reads: Prove that the roots of

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

cannot *all* be real if $2a^2 < 5b$.

The following argument, using calculus, gives a short solution. Let

$$f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e.$$

If all of the roots of the polynomial $f(x)$ are real, then by Rolle's Thm., the same is true of $f'(x)$, $f''(x)$, and $f'''(x)$. But $f'''(x) = 6(10x^2 + 4ax + b)$, and the condition of the problem implies that the discriminant of $f'''(x)$, which is $16a^2 - 40b = 8(2a^2 - 5b)$, is negative. It follows that $f(x)$ cannot have *all* of its roots real.

-- Padmini T. Joshi
Ball State University

Olympiad problems are designed not to require calculus for solution, so solutions here (1/84) did not use it. -- ed.

QUINARY SYSTEM FOR COMPLEX ARITHMETIC

I would like to mention an alternative to the positional number systems in William Gilbert's recent article "Arithmetic in Complex Bases" (this *Magazine*, March 1984, 77-81). This is the *quinary* system with "quint" digits 0, 1, i, $\hat{1}$, $\hat{i}$ ($\hat{a} = -a$) and base $2 + i$. It is comparable to the balanced ternary system for real numbers, which Knuth [1, p. 190] calls "perhaps the prettiest number system of all."

Here are some advantages of this quinary system over systems with base $-n + i$.

1. Negation is trivial:

$$-(a_n \ldots a_1 a_0)_{2+i} = (\hat{a}_n \ldots \hat{a}_1 \hat{a}_0)_{2+i}.$$

Compare this with

$$-(1)_{-1+i} = (11101)_{-1+i} .$$

2. When summing two quinary numerals, the accumulated carry at any column is always one of the nine "carries" in the addition table of Table 1 below. The sum of a carry and two quints is a carry and a quint.

For example, to add $1\hat{1}$, i, and $\hat{1}$ using Table 1, multiply all rows (including the carry row) by i. The $1\hat{1}$ (last column) entry in the i, $\hat{1}$ row (fourth row) is $0i$, 1. Thus

$$1\hat{1} + i + \hat{1} = 0i1 = i1.$$

As a check,

$$[(2 + i) - 1] + i - 1 = i(2 + i) + 1.$$

The addition algorithm represented by Table 1 is intrinsically simpler than the $-n + i$ algorithms described in Gilbert's article. In particular, no clearing routine is necessary. Likewise, multiplication is simpler in the quinary system.

3. As a consequence of the quinary addition algorithm, the sum of two numerals of lengths m and n is a numeral of length at most $\max(m,n) + 2$. This compares with $\max(m,n) + 8$ in base $-1 + i$. The product of two quinary numerals of lengths m and n is a numeral of length at most $m + n + 2$. ($m + n + 8$ in base $-1 + i$?)

I include two pictures, generated by an Apple computer, which indicate the geometry of the quinary system. The first picture shows a "simple" closed curve surrounding all Gaussian integers with quinary numerals of length three or less. The second divides a portion of the plane into regions containing Gaussian integers of numeral lengths 0, 1, ..., 7. Professor Gilbert has pointed out to me that a variant of this latter picture appears in [2, p. 49].

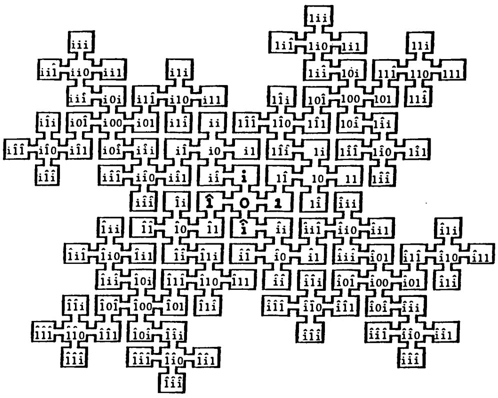
[1] D. E. Knuth, The Art of Computer Programming, Vol. 2/Seminumerical Algorithms, Addison-Wesley, 1981.

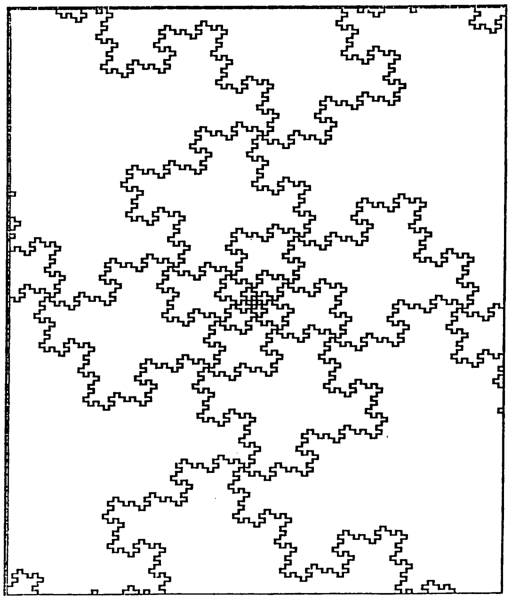
[2] B. B. Mandelbrot, The Fractal Geometry of Nature, Freeman, 1982.

-- Vic Norton
Bowling Green State Univ.

TABLE 1

	QUINT ADDITION TABLE: CARRY + QUINT + QUINT = CARRY QUINT								
QUINT PAIRS	CARRIES 00	01	0i	$0\hat{1}$	$0\hat{i}$	$1\hat{1}$	$i\hat{i}$	$\hat{1}1$	$\hat{i}i$
0 0	00 0	00 1	00 i	00 $\hat{1}$	00 $\hat{i}$	01 $\hat{1}$	0i $\hat{i}$	$0\hat{1}$ 1	$0\hat{i}$ i
0 1	00 1	01 $\hat{i}$	01 $\hat{1}$	00 0	$0\hat{i}$ i	01 0	00 i	00 $\hat{i}$	$\hat{i}i$ $\hat{1}$
1 1	01 $\hat{i}$	$\hat{i}i$ i	01 0	00 1	$\hat{i}i$ $\hat{1}$	01 1	01 $\hat{1}$	$0\hat{i}$ i	$\hat{i}i$ 0
1 i	01 $\hat{1}$	01 0	$1\hat{1}$ $\hat{i}$	00 i	00 1	01 i	0i 1	00 0	01 $\hat{i}$
1 $\hat{1}$	00 0	00 1	00 i	00 $\hat{1}$	00 $\hat{i}$	01 $\hat{1}$	0i $\hat{i}$	$0\hat{1}$ 1	$0\hat{i}$ i





1984 USA MATH OLYMPIAD AND INTERNATIONAL MATH OLYMPIAD

Eight finalists in a mathematics competition involving nearly 400,000 high school students were honored with Olympiad medals on June 5, 1984 in Washington, DC. The final round in this competition was the Thirteenth USA Mathematical Olympiad (USAMO) in which 90 students competed in a challenging examination designed to test ingenuity as well as mathematical background.

The USAMO competitors were the top performers in the American High School Mathematics Examination (AHSME) and the American Invitational Mathematics Examination (AIME) which were held in high schools throughout the United States and Canada in February and March 1984.

The eight USAMO winners are:

*David J. Grabiner	Claremont, CA
*Douglas R. Davidson	McLean, VA
*David J Moews	Willimantic, CT
*Michael Reid	Woodhaven, NY
Joseph G. Keane	Pittsburgh, PA
*Steven Newman	Ann Arbor, MI
William C. Jockusch	Urbana, IL
Andrew Chin	Austin, TX

Following the Awards Ceremony, these winners and sixteen other top performers in the USAMO participated in an intensive three-week seminar at the U.S. Naval Academy at Annapolis. After this training, a U.S. team of 6 students was chosen to travel to Prague, Czechoslovakia on July 4-5, for the 1984 International Mathematical Olympiad (IMO).

The U.S. competitors in the IMO included those USAMO finalists whose names are starred above, and Jeremy Kahn of New York, NY.

Team scores at the IMO were as follows:

1st	USSR	235 points
2nd	Bulgaria	203 points
3rd	Romania	199 points
4th (tie)	Hungary USA	195 points

Individuals on the U.S. team received prizes for their performance: David Moews won a first prize for a score of 42, and David Grabiner, Jeremy Kahn, Steven Newman, and Michael Reid all received second prizes.

The IMO problems will be published in our next issue. We invite you to the challenge of the problems from the 1984 USA and Canadian Math Olympiads, which appear on the next page. Solutions will be published in our next issue.

13th USA MATH OLYMPAD
MAY 1, 1984

1. The product of two of the four roots of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Determine the value of k.

2. The geometric mean of any set of m non-negative numbers is the m-th root of their product.

(i) For which positive integers n is there a finite set S_n of n distinct positive integers such that the geometric mean of any subset of S_n is an integer?

(ii) Is there an infinite set S of distinct positive integers such that the geometric mean of any finite subset of S is an integer?

3. P, A, B, C and D are five distinct points in space such that

$$\angle APB = \angle BPC = \angle CPD = \angle DPA = \theta,$$

where θ is a given acute angle. Determine the greatest and least values of $\angle APC + \angle BPD$.

4. A difficult mathematical competition consisted of a Part I and a Part II with a combined total of 28 problems. Each contestant solved exactly 7 problems altogether. For each pair of problems, there were exactly two contestants who solved both of them. Prove that there was a contestant who in Part I solved either no problems or at least 4 problems.

5. $P(x)$ is a polynomial of degree $3n$ such that

$$P(0) = P(3) = \ldots = P(3n) = 2,$$
$$P(1) = P(4) = \ldots = P(3n-2) = 1,$$
$$P(2) = P(5) = \ldots = P(3n-1) = 0,$$
and $P(3n+1) = 730$.

Determine n.

These problems were set by the USAMO examination subcommittee:

M. S. Klamkin, U. of Alberta, Chairman;
J. Konhauser, Macalester Coll.;
A. Liu, U. of Alberta;
C. C. Rousseau, Memphis St. U.

16th CANADIAN MATH OLYMPIAD
MAY 2, 1984

1. Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

2. Alice and Bob are in a hardware store. The store sells coloured sleeves that fit over keys to distinguish them. The following conversation takes place:

Alice: Are you going to cover your keys?

Bob: I would like to, but there are only 7 colours and I have 8 keys.

Alice: Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.

Bob: You must be careful what you mean by "next to" or "three keys over from" since you can turn the key ring over and the keys are arranged in a circle.

Alice: Even so, you don't need 8 colours.

Problem: What is the smallest number of colours needed to distinguish n keys if all the keys are to be covered.

3. An integer is *digitally divisible* if

(a) none of its digits is zero;
(b) it is divisible by the sum of its digits (e.g., 322 is digitally divisible). Show that there are infinitely many digitally divisible integers.

4. An acute-angled triangle has unit area. Show that there is a point inside the triangle whose distance from each of the vertices is at least

$$\frac{2}{\sqrt[4]{27}}.$$

5. Given any 7 real numbers, prove that there are two of them, say x and y, such that

$$0 \le \frac{x - y}{1 + xy} \le \frac{1}{\sqrt{3}}.$$

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